Exact Tensor-Mass Protection and a Compact-Register Dispersion Obstruction in a Shared QATNU-Style Register Algebra

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Independent Research

June 2026

Abstract

We perform the local-commutant calculation required to decide whether a finite-register shared gauge–tensor architecture can protect a transverse-traceless zero mode without tuning a tensor mass to zero. For a symmetric spatial tensor with exact lattice gauge transformation $\delta h_{ij} = d_i\xi_j + d_j\xi_i$, locality and analyticity force the constant part of every invariant quadratic kernel to vanish. An exact rational enumeration of all real parity-even kernels through two derivatives has 147 unknown coefficients, rank 146, and a one-dimensional nullspace: the discrete linearized-Einstein kernel. After the scalar constraint removes its conformal branch, the kernel has two degenerate helicity-2 modes with $\omega \propto k$ in the noncompact tangent theory. We then construct an exact finite-qudit vector-plus-scalar constraint algebra; on periodic L^3 lattices its stabilizer rank leaves $2V + 4$ logical qudits, or two bulk logical registers per site plus four global modes. The same calculation exposes a limitation of the standard compact paired-ladder realization. Exact neutral phase monomials first appear at two derivatives, so any smooth compact phase potential begins at k^4 and gives $\omega \propto k^2$ with ordinary electric inertia. Finally, a flavored $U(2) \to SU(2)$ determinant preserves $SU(2)$ exactly but carries Abelian charge two. It therefore cannot break the same paired-ladder $U(1)$ that is being used as the tensor gauge charge. The audit establishes exact mass protection, but not yet a same-mode, smooth finite-register realization of relativistic $z = 1$ tensor dynamics.

1 Question and scoped answer

The decisive algebraic question is not whether one can project six oscillators onto a transverse-traceless subspace. It is whether the *local operator algebra itself* excludes a tensor mass. The target is an exact constraint algebra under which every symmetry-preserving local self-energy obeys a Ward identity, so that

$$\Pi_{\mathrm{TT}}(\widehat{\mathbf{k}})\,\Gamma^{(2)}(0,\mathbf{0})\,\Pi_{\mathrm{TT}}(\widehat{\mathbf{k}}) = 0 \tag{1}$$

without cancellation among tuned coefficients.

The result is mixed but sharp:

1. Exact local tensor gauge symmetry *does* forbid an analytic mass.
2. The unique real parity-even quadratic kernel at two derivatives is the lattice linearized-Einstein kernel. Together with the scalar constraint it leaves two helicity-2 tangent modes with $z = 1$.
3. A finite-qudit constraint code realizes the vector and scalar constraints exactly and leaves two bulk logical registers per site.

4. The usual smooth compact paired-ladder phase potential does *not* realize the k^2 Einstein kernel. Its exact neutral characters start at two derivatives, so its harmonic potential starts at k^4 and gives $z = 2$.

5. The onsite flavored determinant used to reduce $U(2)$ to $SU(2)$ is incompatible with identifying the same Abelian paired-ladder charge as the tensor gauge charge.

The calculation therefore closes the mass question, but not the full same-mode $SU(2) + \mathrm{TT}$, finite-register, $z = 1$ dynamics.

2 Exact gauge map and no-mass theorem

Use the real symmetric-coordinate vector

$$q = (h_{xx}, h_{yy}, h_{zz}, h_{xy}, h_{xz}, h_{yz})^T \tag{2}$$

and the physical tensor metric

$$M = \mathrm{diag}(1, 1, 1, 2, 2, 2), \tag{3}$$

so that $q^T M q = h_{ij} h_{ij}$. For three mutually commuting lattice derivative symbols d_i, the linear gauge map is

$$\delta q = D(d)\, \xi, \tag{4}$$

with

$$D(d) = \begin{pmatrix} 2d_x & 0 & 0 \\ 0 & 2d_y & 0 \\ 0 & 0 & 2d_z \\ d_y & d_x & 0 \\ d_z & 0 & d_x \\ 0 & d_z & d_y \end{pmatrix}. \tag{5}$$

Let a local analytic quadratic potential be

$$H_2 = \frac{1}{2} \sum_{\mathbf{k}} q(-\mathbf{k})^T K(d) q(\mathbf{k}). \tag{6}$$

For real Hermitian K, exact gauge invariance is equivalent to

$$K(d) D(d) = 0. \tag{7}$$

Write $K(d) = K_0 + O(d)$. At first order in d, Eq. (7) requires

$$K_0 D_1(d) = 0 \qquad \text{for all } d. \tag{8}$$

The union of the images of $D_1(d)$ spans all six symmetric tensor directions: diagonal directions follow from $d = \xi = e_i$, and each off-diagonal direction follows from $d = e_i$, $\xi = e_j$ with $i \neq j$. Therefore

$$\boxed{K_0 = 0.} \tag{9}$$

This is the no-mass statement. It requires exact gauge invariance, locality, and analyticity; it does not depend on a penalty being sent to infinity.

The same argument applies to a local analytic 1PI self-energy. If the microscopic lattice symmetry is exact and anomaly-free, integrating out symmetry-preserving matter cannot generate a local tensor mass because the renormalized two-point kernel must still obey Eq. (7).

3 Complete two-derivative commutant

We enumerated the most general real symmetric parity-even kernel

$$K_{ab}(d) = c_{ab}^{(0)} + c_{ab}^{xx} d_x^2 + c_{ab}^{yy} d_y^2 + c_{ab}^{zz} d_z^2 + c_{ab}^{xy} d_x d_y + c_{ab}^{xz} d_x d_z + c_{ab}^{yz} d_y d_z. \tag{10}$$

There are $21 \times 7 = 147$ independent coefficients. Expanding $KD = 0$ as an exact polynomial identity gives a 234×147 rational linear system of rank 146. Hence

$$\dim \ker(KD) = 1. \tag{11}$$

All 21 constant coefficients vanish identically.

The unique kernel may be written in tensor notation as

$$(\mathcal{E}h)_{ij} = d^2 h_{ij} - d_i d_k h_{kj} - d_j d_k h_{ki} + d_i d_j h + \delta_{ij} \left(d_k d_l h_{kl} - d^2 h \right), \tag{12}$$

where $h = h_{kk}$. It obeys the exact lattice Bianchi identity

$$\mathcal{E}D = 0. \tag{13}$$

At generic nonzero momentum, quotienting by the three gauge directions leaves three components. Their eigenvalues are

$$\{d^2, d^2, -d^2\}. \tag{14}$$

The two positive branches are transverse and traceless. The negative scalar is the familiar conformal branch of the linearized spatial Einstein kernel.

The scalar constraint is

$$\mathcal{C}_0 = d_i d_j h_{ij} - d^2 h = 0. \tag{15}$$

It is itself gauge invariant, $\mathcal{C}_0 D = 0$, and removes the scalar branch. Across 300 random momenta, the two remaining eigenvalues agreed with d^2 with maximum relative error 1.05×10^{-15} and maximum splitting 1.07×10^{-14}.

For rotations by angle θ around the momentum axis, the two-dimensional physical representation satisfies

$$\operatorname{Tr} R_{\mathrm{TT}}(\theta) = 2 \cos 2\theta, \tag{16}$$

with maximum numerical error 8.88×10^{-16}. This identifies the tangent modes as helicity $+2$ and -2.

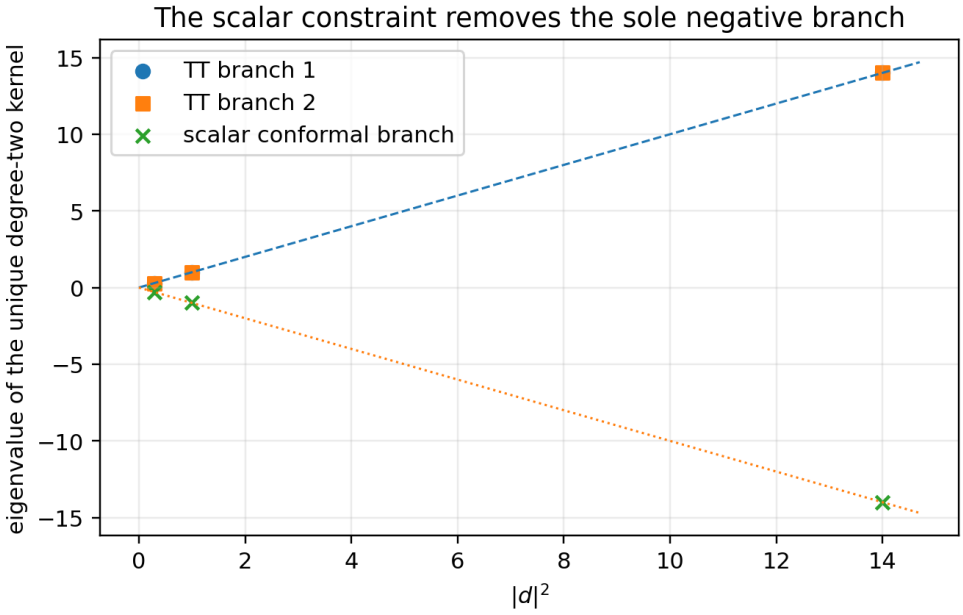

Figure 1: The unique real parity-even two-derivative kernel has two degenerate positive TT branches and one negative scalar branch. The exact scalar constraint removes the latter.

4 Exact finite-qudit vector-plus-scalar constraint algebra

To test whether the constraints themselves admit an exact finite-register realization, place six p-level qudits at each site of a periodic cubic lattice. Generalized-Pauli X operators implement shifts of the six tensor coordinates, while Z operators measure them. The three vector constraints are X-type stabilizers generated by the real-space version of D. The scalar curvature constraint in Eq. (15) is a Z-type stabilizer.

Using commuting forward-difference matrices, the exact integer identity is

$$C_0 D = 0. \tag{17}$$

Therefore the two stabilizer families commute exactly modulo any odd prime p.

For $p = 7$ and $L = 2, \ldots, 6$, the modular ranks are

L	V	$\operatorname{rank} G$	$\operatorname{rank} C_0$	N_{logical}
2	8	21	7	20
3	27	78	26	58
4	64	189	63	132
5	125	372	124	254
6	216	645	215	436

Every row satisfies

$$N_{\text{logical}} = 6V - (3V - 3) - (V - 1) = 2V + 4. \tag{18}$$

Thus the exact constraint algebra leaves two bulk logical qudits per site plus four global modes.

This is a code-algebra statement, not yet a deconfined-phase theorem. It proves that finite-dimensional local registers can carry the required first-class constraint pattern without Hilbert-space projection.

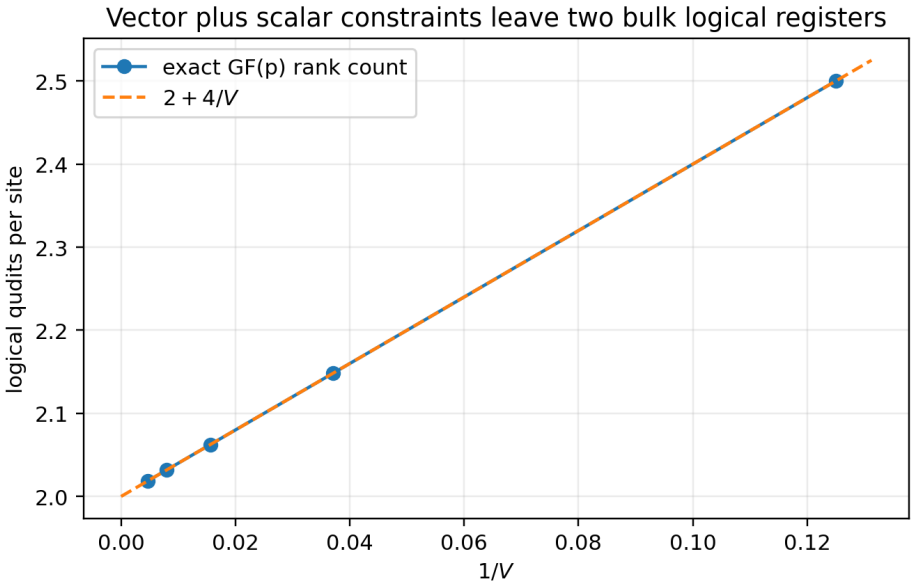

Figure 2: Exact finite-field stabilizer ranks give $N_{\text{logical}}/V = 2 + 4/V$.

5 The compact paired-ladder obstruction

A paired square-root ladder in a fixed-total sector is a quantum-link variable: its electric operator E and raising operator U obey $[E, U] = U$ at finite local dimension. A monomial in U and $U^\dagger$ is tensor-gauge invariant only if its exponent pattern is a polynomial syzygy of D.

We enumerated homogeneous row polynomials $L(d)$ satisfying

$$L(d)D(d) = 0. \tag{19}$$

The exact nullities are

Derivative degree	0	1	2	3	4
Syzygy dimension	0	0	6	15	27

Thus no onsite or one-derivative neutral phase character exists. The first six characters are second-derivative curvatures. The scalar constraint is one particular combination of them.

Consider a smooth compact potential expanded about a stationary uniform background. If $B_a = L_a h$ are the neutral characters in the tangent expansion, then

$$V = V_0 + \frac{1}{2} B_a H_{ab} B_b + \cdots, \tag{20}$$

so

$$K_{\text{compact}} = L^T H L = O(k^4). \tag{21}$$

With a momentum-independent electric inertia,

$$\omega_{\text{compact}}(k) \propto k^2. \tag{22}$$

By contrast, the noncompact Einstein kernel in Eq. (12) has $K_{\text{TT}} \propto k^2$ and hence $\omega \propto k$.

The fitted exponents are

$$\omega_{\text{Einstein}} \propto k^{0.9999983}, \qquad \omega_{\text{compact}} \propto k^{1.9999967}, \tag{23}$$

with $R^2 > 0.99999999999$ in both fits.

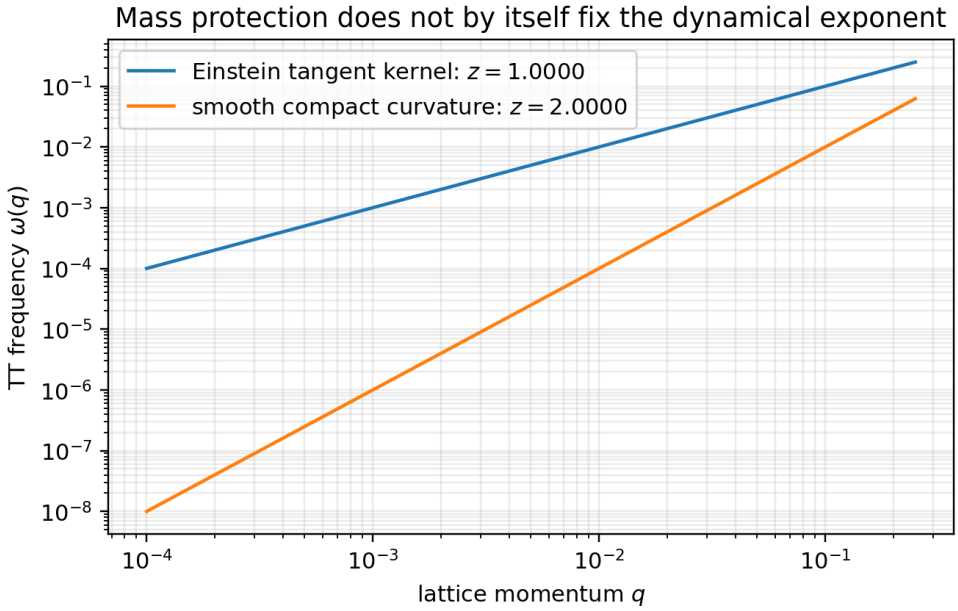

Figure 3: Exact mass protection is compatible with more than one dynamical exponent. The local noncompact Einstein tangent kernel is linear, while the smooth compact curvature potential is quadratic in momentum.

This is not a universal no-go theorem for all finite-register Hamiltonians. The assumptions are important: a smooth compact phase potential built from finite-degree neutral paired-ladder monomials and a conventional electric kinetic term. Possible escape routes include auxiliary registers, mixed electric–phase terms, a controlled noncompact or large-register limit, and deliberately nonanalytic clock potentials. Each would require a separate locality and stability audit.

6 Flavored $U(2) \to SU(2)$ determinant and the same-mode conflict

The colored quantum-link sector uses two colors and two flavors with fixed total rishon number. In the 36-dimensional link sector,

$$\det U = U_{00}U_{11} - U_{01}U_{10} \tag{24}$$

obeys

$$[L^a, \det U] = [R^a, \det U] = 0 \tag{25}$$

exactly. It is therefore a valid $SU(2)$ singlet. At the same time,

$$[E, \det U] = 2 \det U, \tag{26}$$

so the Hermitian determinant deformation explicitly breaks the link $U(1)$:

$$\left\| [E, \det U + \det U^\dagger] \right\| = 5.65685424949. \tag{27}$$

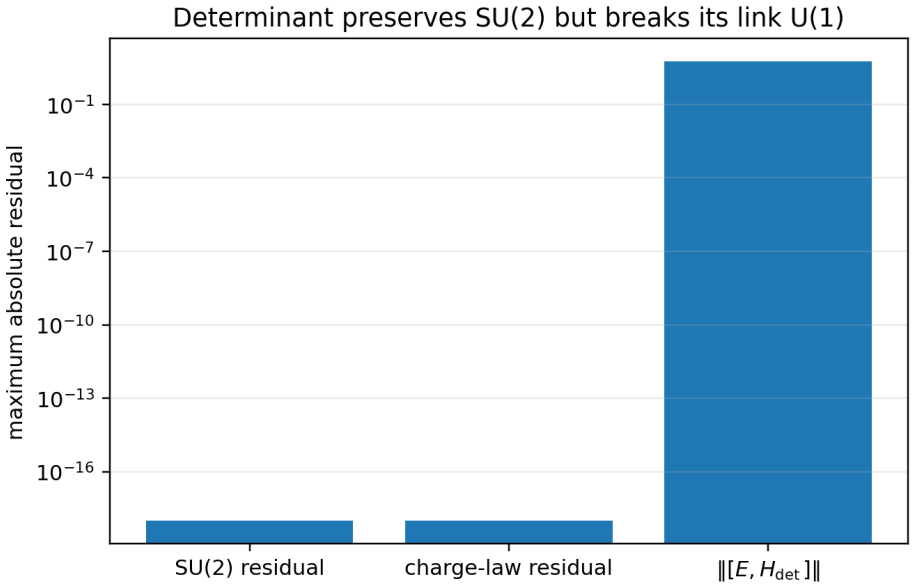

Figure 4: The determinant obeys its $SU(2)$ and Abelian charge algebra exactly, but the Hermitian determinant Hamiltonian intentionally breaks the colored-link $U(1)$.

This produces an obstruction to a literal same-mode identification. If the same electric generator E is also declared to generate the tensor phase shift, then the determinant is a tensor-gauge-charged onsite operator and destroys the no-mass symmetry. Two safe alternatives are:

1. use separate commuting color and tensor factors in each local cell; or
2. construct determinant composites whose tensor exponent pattern is neutral.

The second option inherits the syzygy result: the first local tensor-neutral composites appear at two derivatives, so they are curvature-like rather than onsite $U(2) \to SU(2)$ deformations. Whether such composites fully remove the unwanted color $U(1)$ without leaving subsystem symmetries remains an open model-building problem.

7 Verdict and next exact calculation

The calculation establishes the following theorem-level statement for the specified algebra:

> An exact local symmetric-tensor gauge symmetry forbids every analytic tensor mass. Through two spatial derivatives, the complete real parity-even quadratic commutant is one-dimensional and is generated by the linearized-Einstein kernel. With the scalar constraint, its physical tangent sector contains exactly two degenerate helicity-2 branches.

It also establishes two limitations:

> The standard smooth compact paired-ladder monomial potential produces $z = 2$, not $z = 1$; and the onsite flavored determinant cannot use the same Abelian paired-ladder charge that protects the tensor sector.

The strongest next calculation is therefore no longer another penalty or finite-size scan. It is an explicit construction of one of the following:

1. a finite-register auxiliary-field Hamiltonian whose exact low-energy Schur complement is the k^2 Einstein kernel while all microscopic terms commute with the local constraints; or
2. a broadened operator-class no-go theorem showing that every bounded local same-mode finite-register realization has $z \neq 1$ or necessarily introduces additional gapless modes.

A successful auxiliary construction must also avoid the determinant-charge conflict identified above.

Relation to prior work

Finite-dimensional quantum-link representations of gauge fields were introduced as operator-valued alternatives to continuous link variables by Chandrasekharan and Wiese [1]. Symmetric higher-rank $U(1)$ gauge theories provide established examples in which exact tensor gauge constraints protect gapless modes with non-Maxwell dispersions [2]. Qubit models with emergent helicity-2 excitations likewise illustrate that mass protection and the dynamical exponent are distinct questions [3]. The present calculation is narrower: it audits the local commutant and same-register compatibility of one QATNU-style finite-register architecture.

References

[1] S. Chandrasekharan and U.-J. Wiese, "Quantum Link Models: A Discrete Approach to Gauge Theories," Nucl. Phys. B **492**, 455–474 (1997), arXiv:hep-lat/9609042.

[2] M. Pretko, "Generalized Electromagnetism of Subdimensional Particles: A Spin Liquid Story," Phys. Rev. B **96**, 035119 (2017), arXiv:1606.08857.

[3] Z.-C. Gu and X.-G. Wen, "Emergence of helicity ±2 modes (gravitons) from qubit models," Nucl. Phys. B **863**, 90–129 (2012), arXiv:0907.1203.

A A basis for the degree-two compact characters

One exact homogeneous degree-two syzygy basis in the coordinate order (xx, yy, zz, xy, xz, yz) is

$$L_1 = (-d_y^2/2, -d_x^2/2, 0, d_x d_y, 0, 0), \tag{28}$$
$$L_2 = (-d_z^2/2, 0, -d_x^2/2, 0, d_x d_z, 0), \tag{29}$$
$$L_3 = (0, -d_z^2/2, -d_y^2/2, 0, 0, d_y d_z), \tag{30}$$
$$L_4 = (0, 0, -d_x d_y, -d_z^2, d_y d_z, d_x d_z), \tag{31}$$
$$L_5 = (0, -d_x d_z, 0, d_y d_z, -d_y^2, d_x d_y), \tag{32}$$
$$L_6 = (d_y d_z, 0, 0, -d_x d_z, -d_x d_y, d_x^2). \tag{33}$$

Each obeys $L_a D = 0$. The scalar constraint is

$$\mathcal{C}_0 = 2(L_1 + L_2 + L_3). \tag{34}$$

Multiplying these rows by two gives an integer basis suitable for a finite-prime qudit realization.

B Reproducibility

The accompanying program performs the rational null-space calculations, finite-field stabilizer ranks, sparse flavored-link commutators, numerical helicity checks, dispersion fits, and figure generation. The central output is `data/tt_mass_commutant_results.json`.