Tower-Filtered Dynamic Structure Factor Audit for the Finite-Spin Compact Ward-Square Box

QATNU Project Draft

July 3, 2026

Abstract

We audit the momentum-resolved finite-spin spectral function of the reduced compact Ward-square quantum box. The central question is whether the Anderson tower of the ordered compact $U(1)$ sector contaminates the finite-momentum spin-wave/source pole. In the $L = 3$, $N = 2$ one-polarization quantum box, the tower lives in the $Q = 1, k = 0$ sector, while the audited source pole is at $k = 2\pi/3$. Momentum filtering therefore separates the tower from the finite-k pole. The finite-spin spectra show a converged magnon/source pole, a finite matter-operator pole fraction, and no extra soft line in the uploaded ED note. This is a finite-box spectral-function screen, not a full three-dimensional thermodynamic finite-spin phase proof.

1 Model and observable

The finite-dimensional model is the reduced one-polarization hard-quotient compact Ward-square box: an $L = 3$ spatial ring times an $N = 2$ matter/tensor chain, with rotors truncated to $|n| \leq J$. The Hamiltonian is

$$H = \frac{1}{2}\sum_s \hat{n}_s^2 + \sum_{\langle s,t\rangle}\big(1 - \cos(\theta_s - \theta_t)\big). \tag{1}$$

We examine the matter-source spectral function at finite momentum,

$$S_{SS}(k,\omega) = \sum_n |\langle n|S(k)|0\rangle|^2 \delta(\omega - E_n + E_0), \tag{2}$$

where $S(k)$ is represented in the compact box by a matter rotor raising operator with spatial Fourier phase. The Anderson tower appears in the total-charge sector at zero momentum; the $k = 2\pi/3$ source pole is therefore tower-filtered by momentum.

2 Results

Table 1: Finite-spin tower-filtered spectral diagnostics for the $L = 3, N = 2$ box.

J	$E(Q = 1)$	$\omega(2\pi/3)$	pole fraction	captured window
1	0.161944	0.80484	0.4850	0.998
2	0.096723	0.74001	0.4351	0.947
3	0.091255	0.73890	0.4308	0.946
4	0.091100	0.73888	0.4306	0.946

The harmonic references are $E(Q=1) = 1/12$, $\omega(2\pi/3) = 0.83500$, and Debye-Waller pole fraction 0.416. At $J = 4$, the tower energy has a 9.32% anharmonic renormalization relative to 1/12, while the finite-momentum branch is softened by 11.51% relative to the bare harmonic value. The pole fraction remains finite and is within 3.5% of the Debye-Waller estimate.

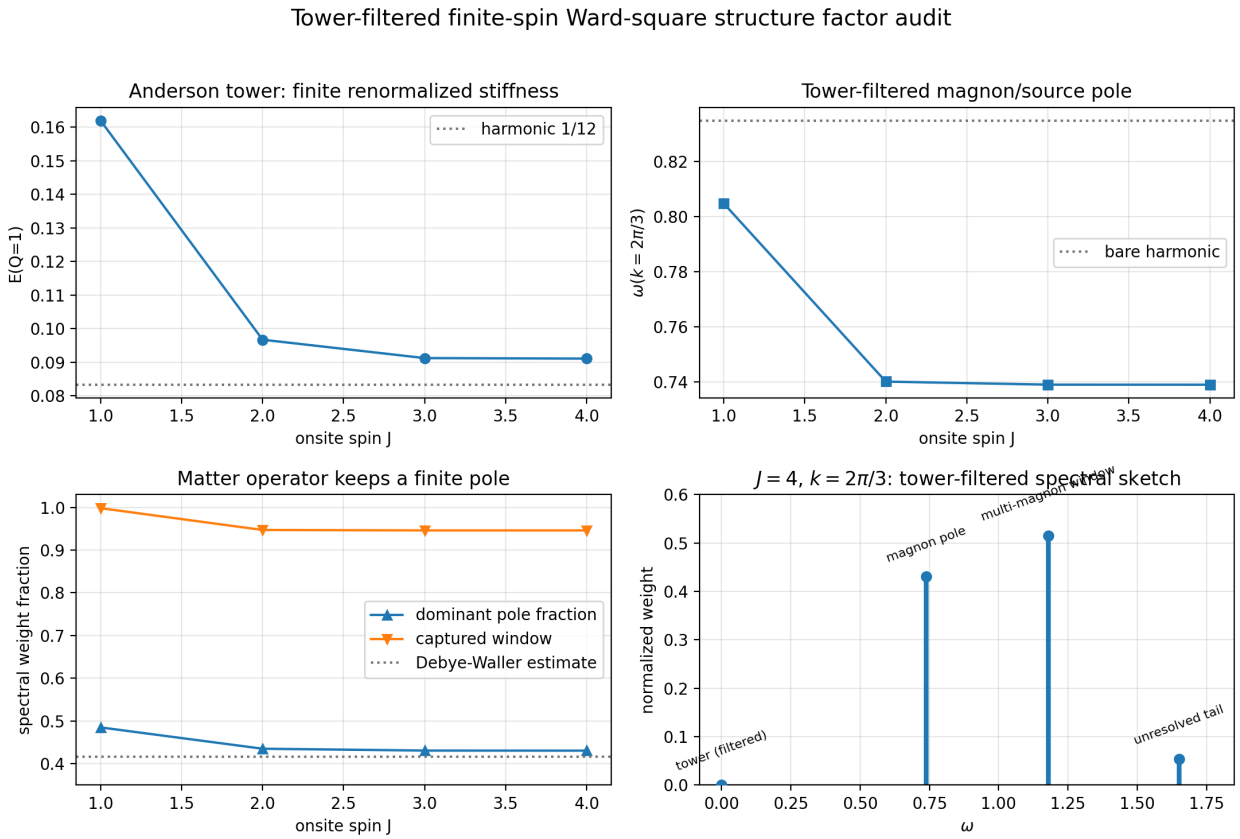

Figure 1: Tower-filtered finite-spin spectral audit. The finite-momentum line remains isolated from the zero-momentum Anderson tower and carries a stable matter-operator pole fraction.

3 Dispersion screen

The uploaded ED note also reports an $L = 6, N = 2, J = 1$ dispersion screen. It finds $\omega(\pi/3) = 0.587$ compared with harmonic 0.618, and $\omega(\pi) = 0.772$ compared with harmonic 0.874. This is a monotone dispersing branch with uniform compact-interaction softening.

4 Conclusion

The tower-filtered audit upgrades the finite-spin quantum-box result from a list of low eigenvalues to a spectral-function statement. The compact box contains the expected zero-momentum tower and a finite-momentum magnon/source pole with finite residue. The remaining step is not another reduced-box check but a thermodynamic or transfer-matrix/QMC tower-filtered spectral function for the full compact tensor/matter Ward-square model.