Three-Dimensional Local Logical Pairing and a Finite-Range Compact No-Go

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Abstract

We perform the calculation left open by the bounded clock/Villain study of a six-component tensor register with six auxiliary registers. First, we correct the previous radius-two locality count: the reported twelve Z and fourteen X logical operators were artifacts of using a periodic $L = 5$ torus whose radius-two stencils wrap around. An infinite-patch audit embedded independently in $L = 15$ and $L = 17$ lattices gives five radius-two Z logical generators and five radius-two X logical generators. Their same-center symplectic pairing vanishes, while their translation-covariant Fourier pairing has physical rank two at every nonzero derivative momentum and is homogeneous of degree three. Hence no bounded-range polynomial change of variables can convert them into onsite canonical pairs. Exact rational syzygy enumeration shows that physical compact Z characters first occur at second derivative order and physical X characters at first derivative order. The most general analytic positive finite-range Hessian therefore has blocks $H_{pp} = O(k^2)$, $H_{pq} = O(k^3)$, and $H_{qq} = O(k^4)$. No positive k^2 transverse-traceless potential exists in this exact compact centralizer class; generic stable frequencies scale as $\omega \sim k^3$. We also identify a Brillouin-edge doubler in the central-difference regulator and verify that a forward-difference control removes the doubler without changing the infrared obstruction. The result is a scoped no-go for the present exact CSS/clock construction, not a no-go for every finite-register tensor architecture.

1 Question and scope

The preceding bounded-register calculation found two exact local transverse-traceless clock variables after fixing a propagation axis, but the full three-dimensional locality audit found no nontrivial physical logical operator at radius zero or one. Radius-two logical spaces appeared, leaving a decisive question:

> Can the radius-two logicals be assembled into translation-covariant bounded-range canonical variables, and can a stable exact compact Hamiltonian built from them generate a positive k^2 TT kernel?

The result below answers both questions negatively for a precise class: translation-invariant finite-range finite-clock or Pauli-character Hamiltonians that commute exactly with the CSS constraint algebra and possess an analytic positive quadratic expansion around a stable minimum. The statement does not cover nonlocal inverse derivatives, gapless auxiliaries, only-dressed symmetries, or a different microscopic constraint algebra.

2 Constraint code and corrected locality audit

Each site contains twelve clock registers: six components of a symmetric tensor h_{ij} and six auxiliary components y_A. The X-type gauge map is

$$D_{\mathrm{ext}} = \begin{pmatrix} D \\ -\frac{1}{2}LD \end{pmatrix}, \tag{1}$$

where $D\xi$ is the lattice symmetrized gradient and L is the first-order divergence/trace-gradient map. The Z stabilizers are

$$U = \begin{pmatrix} \frac{1}{2}L & I_6 \end{pmatrix}, \qquad C_0 = \begin{pmatrix} R^L & 0 \end{pmatrix}, \tag{2}$$

with $R^L h = d_i d_j h_{ij} - d^2 h_{ii}$. They commute exactly because $UD_{\mathrm{ext}} = 0$ and $R^L D = 0$.

The earlier locality calculation used an $L = 5$ periodic torus. Both the test operator and the extended constraint stencil reach Manhattan radius two. Their commutator equations therefore extend beyond half the torus and wrap

Table 1: Corrected central-stencil local quotient dimensions.

R	sites	Z_{comm}	Z_{triv}	$Z_{\log}$	X_{comm}	X_{triv}	$X_{\log}$
0	1	0	0	0	0	0	0
1	7	10	10	0	0	0	0
2	25	71	66	**5**	8	3	**5**
3	63	255	223	32	58	21	37

around, producing spurious local relations. Replacing the torus calculation by an infinite-patch calculation embedded in larger lattices gives Table 1; $L = 15$ and $L = 17$ agree exactly.

The old wrapped $L = 5$ count was $(12, 14)$ at radius two. Figure 1 shows the correction graphically.

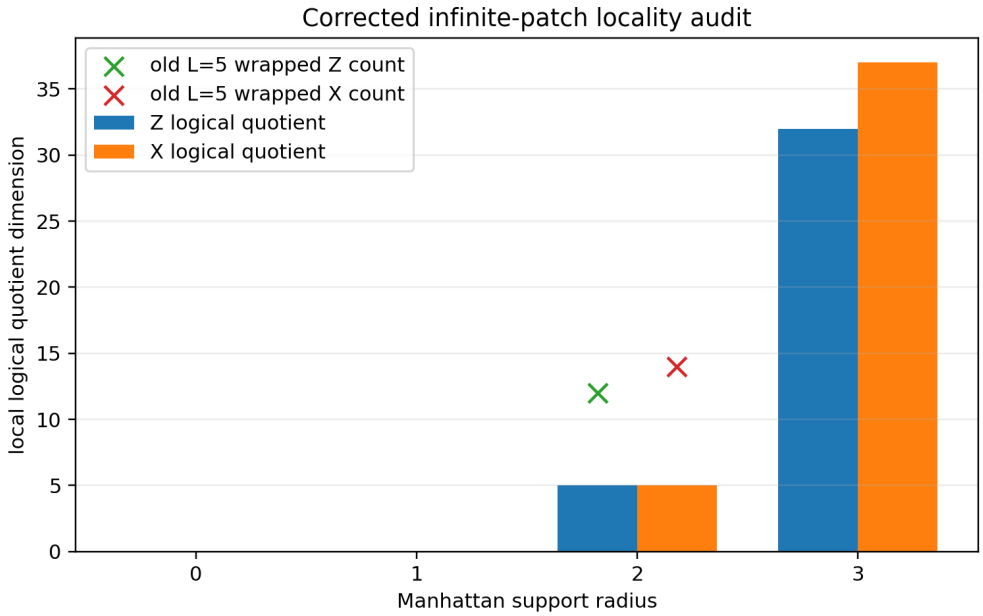

Figure 1: The periodic $L = 5$ radius-two count is not an infinite-lattice locality result. The corrected radius-two quotient is five dimensional in each CSS sector.

3 Exact minimal logical modules

The local dimensions have a clean polynomial interpretation. The auxiliary stabilizers contain an identity on y, so any physical Z character can be reduced modulo stabilizers to an h-only row $z(\boldsymbol{d})$ obeying

$$z(\boldsymbol{d})D(\boldsymbol{d}) = 0. \tag{3}$$

An exact rational homogeneous syzygy calculation gives nullity zero at degrees zero and one, and nullity six at degree two. One of those six rows is the scalar stabilizer R^L, leaving five degree-two Z logical generators.

For X characters, $Ux = 0$ fixes the auxiliary component in terms of the h component, while the scalar constraint imposes

$$C(\boldsymbol{d})x(\boldsymbol{d}) = 0. \tag{4}$$

There is no constant solution. At degree one there are eight solutions; three are the gauge columns of D, leaving five degree-one X logical generators. Thus

$$Z_a(\boldsymbol{d}) = O(d^2), \qquad X_b(\boldsymbol{d}) = O(d). \tag{5}$$

These dimensions reproduce the corrected finite-field radius-two quotient exactly.

4 The translation-covariant symplectic form

For exact polynomial bases $Z(\boldsymbol{d})$ and $X(\boldsymbol{d})$, define

$$\Omega(\boldsymbol{d}) = Z(\boldsymbol{d})X(\boldsymbol{d})^T. \tag{6}$$

The same-center pairing of the extracted finite-field radius-two bases is identically zero. Pairing appears only after translation. Over all 216 nonzero triples in $\mathrm{GF}(7)^3$, every one of the 208 points with nonzero central derivative has

$$\dim Z_{\mathrm{phys}} = 2, \qquad \dim X_{\mathrm{phys}} = 2, \qquad \operatorname{rank}\Omega = 2. \tag{7}$$

The remaining eight points are zeros of the central derivative.

The exact real polynomial matrix Ω is homogeneous of degree three and has generic rank two. A basis-independent certificate is the sum of squared two-by-two minors,

$$\begin{aligned}\sum_{I,J} \det \Omega_{I,J}^2 = \frac{1}{4}(d_x^2 + d_y^2 + 2d_z^2)(d_x^2 - d_x d_y + d_y^2 + d_z^2) \\ \times (d_x^2 + d_x d_y + d_y^2 + d_z^2)P_6(\boldsymbol{d}),\end{aligned} \tag{8}$$

where P_6 has positive coefficients in all its even monomials. Hence this expression is positive for every real $\boldsymbol{d} \neq 0$ and the rank is exactly two in every nonzero direction.

Homogeneity gives

$$\Omega(\lambda \boldsymbol{d}) = \lambda^3 \Omega(\boldsymbol{d}), \qquad \sigma_{1,2}(\Omega) \asymp |\boldsymbol{d}|^3, \tag{9}$$

as verified in Fig. 2.

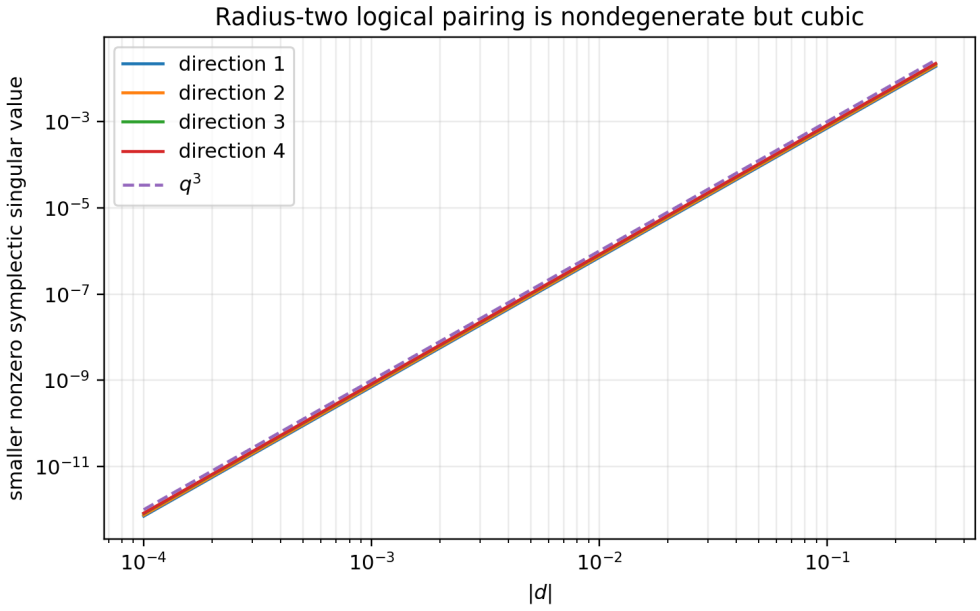

Figure 2: The two nonzero symplectic singular values vanish cubically. The five local generators span the correct two-dimensional physical quotient, but not an onsite canonical algebra.

A finite-range translation-covariant canonicalization would require Laurent-polynomial matrices A and B regular at $\boldsymbol{d} = 0$ such that

$$A(\boldsymbol{d})\Omega(\boldsymbol{d})B(\boldsymbol{d})^T = I_2. \tag{10}$$

The left side belongs to the third power of the momentum ideal and vanishes at the origin; the right side does not. Therefore no bounded-range canonicalization exists. Any inverse must diverge as $|d|^{-3}$ and is nonlocal.

5 Complete leading-order compact Hamiltonian screen

A finite-clock Hamiltonian has an exact Pauli/character expansion. Conjugation by a stabilizer acts diagonally on that basis, so exact commutation permits only centralizer characters. After quotienting stabilizers and gauge directions, Eq. (5) fixes the lowest possible derivative orders of every physical Fourier component.

The most general analytic positive quadratic expansion on the two TT coordinates has the form

$$H_2(\boldsymbol{k}) = \frac{1}{2}\begin{pmatrix} q \\ p \end{pmatrix}^T \begin{pmatrix} B(\boldsymbol{k}) & C(\boldsymbol{k})^T \\ C(\boldsymbol{k}) & A(\boldsymbol{k}) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \tag{11}$$

where

$$A = O(k^2), \qquad C = O(k^3), \qquad B = O(k^4). \tag{12}$$

No analytic coefficient choice can produce a k^2 term in B. Integrating a gapped block does not change the conclusion:

$$B_{\mathrm{Schur}} = B - C^T A^{-1} C = O(k^4). \tag{13}$$

Thus the positive Einstein-order kernel is absent from this compact character cone.

For generic positive leading coefficients, the symplectic eigenvalues satisfy

$$\omega^2 \sim AB - C^2 = O(k^6), \qquad \boxed{\omega \sim k^3}. \tag{14}$$

Figures 3 and 4 show one deterministic positive representative.

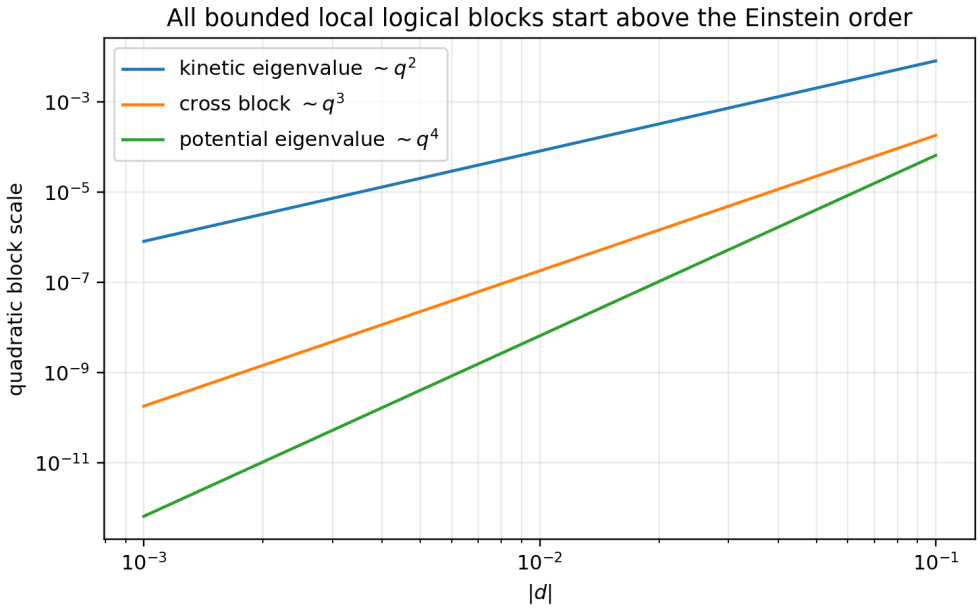

Figure 3: Exact homogeneity of the kinetic, mixed, and potential blocks. The desired k^2 TT potential is not present.

As a broad consistency check, one hundred random positive 10×10 leading Hessians were tested in three momentum directions. The resulting six hundred branch fits satisfy

$$2.999999999999998 \le z \le 3.000000000000002, \tag{15}$$

with $R^2 = 1$ at displayed precision. This numerical scan checks the implementation; exact homogeneity is the proof.

6 A separate stencil defect

The central derivative used in the earlier finite-field implementation has symbol $d(k) \propto \sin k$. It vanishes both at $k = 0$ and at $k = \pi$, introducing Brillouin-edge tensor doublers. A forward derivative has $|d(k)| = 2|\sin(k/2)|$ and removes the edge zero. The corresponding full-zone comparison is included in the reproducibility archive.

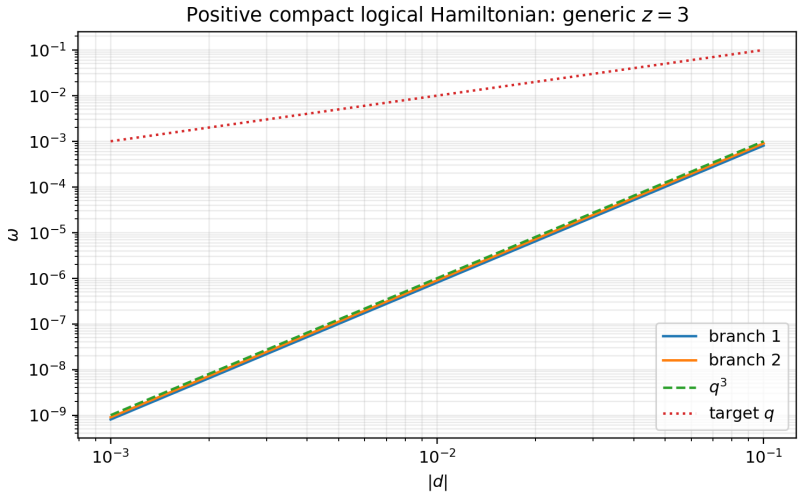

Figure 4: Both physical branches scale as k^3 for a positive compact logical Hessian. The target relativistic k line is parametrically separated.

The forward-difference infinite-patch audit also has no radius-zero or radius-one logicals. Its raw radius-two quotient is larger because an asymmetric radius-two ball contains multiple translates of the minimal homogeneous generators; the exact low-degree module and the infrared conclusion are unchanged.

7 Verdict and surviving routes

The decisive result is

> **The corrected three-dimensional local logicals are physically complete at nonzero momentum, but their symplectic form vanishes cubically. They do not admit bounded-range canonical pairing. Every stable exact compact character potential starts at k^4, and generic complete compact dynamics scales as $\omega \sim k^3$, not $\omega \sim k$.**

This result is narrower than a universal no-go. It does not invalidate the exact noncompact Einstein kernel or the auxiliary Schur parent. Instead, it shows that those structures do not descend to a generic three-dimensional bounded exact-clock Hamiltonian merely by replacing the fields with local physical logical operators.

The remaining viable routes are structurally different:

1. construct a bounded non-character/Villain term that realizes the auxiliary Schur parent while maintaining exact locality and stability;
2. allow a gapless auxiliary sector whose elimination is nonanalytic;
3. seek a quasi-locally dressed rather than exact stabilizer symmetry under an independently chosen perturbation;
4. replace the microscopic tensor constraint algebra so that local canonical TT variables exist.

Another search over local logical Hamiltonians in the present CSS code cannot produce the missing k^2 kernel.

Reproducibility

The archive contains the complete GF(7) bases, exact symbolic generators and pairing matrix, random positive-Hessian scans, forward-stencil control, figures, and a clean reproduction script. The main result file has SHA-256 digest

`cd4d7124a0aa2b63d0dd9890d41ebef1c41c53370b8a72e14bf7e0a4392ad9d4`.