Unbounded Photon-Number Support and Continuum Limits for Finite Quantum Controllers

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Boundary changes can turn a one-quantum state into a photon-added Gaussian state whose reduced number distribution extends without an occupation cutoff. We prove a basis-fixed support theorem for this mechanism. Let a normalizable pure squeezed state be written as $|G_Z\rangle \propto \exp[\frac{1}{2}a^\dagger Z a^\dagger]|0\rangle$ on a bipartition $A \oplus B$. If the cross block Z_{AB} is nonzero and the added creation operator has a nonzero component in the discarded sector B, then the retained distribution satisfies $P_A(n) > 0$ for every finite $n \geq 0$. The proof works directly in the physical mode basis and allows arbitrary same-side squeezing. We distinguish this normalizable unbounded tail from the ideal abrupt-boundary limit: for the finite-box shutter coefficients, $\mathrm{Tr}(\beta_K \beta_K^\dagger) = (2\pi^2)^{-1}(\log K)^2 + O(\log K)$, so the limiting Bogoliubov map is not implementable on the original Fock space. We then establish a continuum theorem for a fixed finite-dimensional quantum controller coupled autonomously to a bosonic field through finitely many positive smeared quadratic forms. Weighted form-factor convergence implies norm-resolvent convergence of the Hamiltonians, strong convergence of the unitary dynamics uniformly on compact time intervals, and trace-norm convergence of full and reduced density operators. A smooth shutter regulator and a finite-mode controller calculation provide numerical audits. The results close two mathematical questions for smooth autonomous shutters; they do not cover a pointlike perfect boundary or an infinitely abrupt switch.

I. INTRODUCTION

Time-dependent boundaries and media mix positive- and negative-frequency field modes. The resulting Bogoliubov transformation is the standard mechanism behind moving-mirror radiation and the dynamical Casimir effect[1–3]. A recent theoretical shutter construction sharpens the associated locality question: a truncated one-photon wave packet can be locally equivalent to a photon on one side and vacuum on the other while its global state contains arbitrarily high particle-number sectors[4].

Two mathematically different statements are easily conflated. First, a normalizable Fock state may have an *unbounded* number distribution,

$$P(n) > 0 \quad \text{for every finite } n, \tag{1}$$

while retaining finite norm and possibly finite mean number. Second, an ideal boundary quench may have a non-Hilbert–Schmidt anomalous Bogoliubov block, so that the in/out transformation is not unitarily implementable in the original Fock representation[5,6]. The latter is not a normalized "outcome at $n = \infty$."

A separate issue arises when the boundary actuator is quantum. A finite controller can be incorporated into a fixed, time-independent Hamiltonian, but a continuum claim requires control of the ultraviolet regulator and of the reduced controller dynamics. Self-adjointness by itself is insufficient; one needs a topology in which the regulated Hamiltonians, unitaries, and reduced states converge.

This article proves two results. Section II gives a direct physical-mode theorem for the full retained-number support of photon-added pure Gaussian states. Section III derives the abrupt finite-box divergence and states its representation-theoretic meaning. Section IV proves norm-resolvent and dynamical convergence for a fixed finite quantum controller coupled through positive smeared quadratic field forms. Sections V and VI provide numerical audits and delimit the claims. The Gaussian and operator-theoretic ingredients are standard[7–10];

the contribution is the support theorem in a fixed outgoing basis and the explicit continuum closure for an autonomous finite controller.

II. FULL RETAINED-NUMBER SUPPORT

A. Photon-added squeezed states

Let the one-particle space split as

$$\mathfrak{h} = \mathfrak{h}_A \oplus \mathfrak{h}_B, \tag{2}$$

with bosonic Fock factorization $\mathcal{F}_s(\mathfrak{h}) = \mathcal{F}_s(\mathfrak{h}_A) \otimes \mathcal{F}_s(\mathfrak{h}_B)$. Let $Z : \overline{\mathfrak{h}} \to \mathfrak{h}$ be complex symmetric, Hilbert–Schmidt, and satisfy $\|Z\| < 1$. Then

$$|G_Z\rangle = \det(\mathbb{I} - ZZ^\dagger)^{1/4} \exp\left(\frac{1}{2} a^\dagger Z a^\dagger\right) |0\rangle \tag{3}$$

is a normalized zero-mean pure Gaussian vector. The determinant is the Fredholm determinant, well defined because $ZZ^\dagger$ is trace class. In the fixed $A \oplus B$ basis write

$$Z = \begin{pmatrix} Z_{AA} & Z_{AB} \\ Z_{AB}^{\mathsf{T}} & Z_{BB} \end{pmatrix}, \qquad f = f_A \oplus f_B. \tag{4}$$

The normalized photon-added state is

$$|\Psi\rangle = \frac{a^\dagger(f)|G_Z\rangle}{\|a^\dagger(f)|G_Z\rangle\|}. \tag{5}$$

Let $N_A = \mathrm{d}\Gamma(\mathbb{I}_{\mathfrak{h}_A})$, let P_n^A be its n-particle spectral projection, and define

$$P_A(n) = \langle \Psi | \mathsf{P}_n^A | \Psi \rangle. \tag{6}$$

Theorem 1 (Unbounded support in the fixed mode basis)**.** *If*

$$Z_{AB} \neq 0, \qquad f_B \neq 0, \tag{7}$$

then

$$\boxed{P_A(n) > 0 \quad \textit{for every } n = 0, 1, 2, \dots.} \tag{8}$$

No occupation cutoff is introduced. The same-side blocks Z_{AA} and Z_{BB} are unrestricted subject to the normalizability assumptions above.

Proof. Use the Bargmann–Fock representation[11]. Up to a nonzero normalization, the holomorphic wave function of Eq. (5) is

$$F(z_A, z_B) = \big[\ell_A(z_A) + \ell_B(z_B)\big] \exp\Bigg[\frac{1}{2} A(z_A, z_A) + C(z_A, z_B) + \frac{1}{2} B(z_B, z_B)\Bigg], \tag{9}$$

where C represents Z_{AB} and $\ell_B \neq 0$ represents f_B. Because $C \neq 0$, choose $u \in \mathfrak{h}_A$ such that the linear form $d(z_B) = C(u, z_B)$ is nonzero. Restrict $z_A = tu$ and divide by the nonvanishing factor $\exp[B(z_B, z_B)/2]$. The coefficient of t^n is

$$\frac{\ell_B(z_B) d(z_B)^n}{n!} + R_n(z_B), \tag{10}$$

where R_n has polynomial degree at most $n-1$ in z_B. The first term has degree $n+1$ and is nonzero because it is the product of two nonzero linear forms. It therefore cannot be canceled by R_n. Hence the component of F homogeneous of degree n in z_A is nonzero. That component is precisely $\mathsf{P}_n^A|\Psi\rangle$, so its squared norm is strictly positive. The argument uses only the finite-dimensional subspace spanned by u, f_B, and the Riesz vector of d; it therefore applies unchanged in separable infinite dimension. □

The two hypotheses have transparent roles. If $f_B = 0$, then $P_A(0) = 0$ because the added quantum is certainly retained. If $Z_{AB} = 0$, the two sectors are not pair-correlated and an unbounded retained tail need not occur. Theorem 1 is basis fixed: it refers to the same outgoing A/B modes in which Z and f are specified, not to a number operator introduced only after local active Gaussian transformations.

B. One-pair distribution

For one two-mode squeezed pair, define $\lambda = \tanh^2 r$ and

$$|\Psi\rangle = \sqrt{p}\, S_2(r)|1,0\rangle + e^{\mathrm{i}\phi}\sqrt{1-p}\, S_2(r)|0,1\rangle. \tag{11}$$

The two branches have different number difference and are orthogonal. Tracing the second mode gives

$$\boxed{P(k) = (1-\lambda)^2\big[(1-p)(k+1)\lambda^k + p\,k\lambda^{k-1}\mathbf{1}_{k\geq 1}\big].} \tag{12}$$

For $0 < \lambda < 1$ and $0 < p < 1$, the distribution is normalized and strictly positive at every finite k. Its tail and mean are

$$P(k) \sim Ck\lambda^k, \qquad \langle k\rangle = \frac{p + \lambda(2-p)}{1-\lambda}. \tag{13}$$

Thus "from zero to infinity" means support on arbitrarily large finite values, not an event labeled by infinite particle number.

C. Smooth-shutter certificate

We evaluated the hypotheses in a smooth doubled-chiral lattice regulator with 128 forward and 128 backward modes. The transition width is 0.35 in a box of length 8, with mass parameter 0.8. The canonical squeezed matrix obeys

$$\|Z\| = 0.3963231 < 1, \qquad \|Z_{AB}\|_{\mathrm{F}} = 0.4689278, \tag{14}$$

while the added mode has

$$\|f_A\|^2 = 0.7452284, \qquad \|f_B\|^2 = 0.2512941. \tag{15}$$

The anomalous block is Hilbert–Schmidt with $\mathrm{Tr}(\beta\beta^\dagger) = 0.7790047$, and the reduced covariance contains 14 nontrivial Gaussian Schmidt pairs. Hence Theorem 1 applies to the regulated outgoing modes without a Fock occupation ceiling.

Figure 1(a) plots Eq. (12) using the largest measured pair parameter, $\lambda = 0.1124749$, and an illustrative branch weight drawn from Eq. (15); it is not presented as the complete multimode shutter distribution. Panel (b) displays the nonzero pair spectrum that certifies cross-sector squeezing.

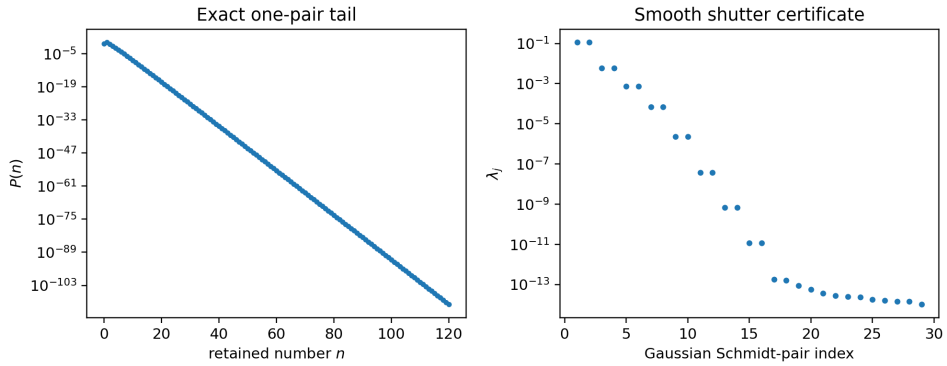

FIG. 1. Unbounded-support certificate. (a) Exact one-pair distribution, illustrating strict support at every finite number. For the displayed parameters, $P(100) = 6.93 \times 10^{-93}$. (b) Nonzero Gaussian pair parameters of the smooth $128 + 128$ shutter regulator. Together with $\|f_B\| > 0$, the nonzero cross block invokes Theorem 1.

III. ABRUPT PERFECT BOUNDARIES AND LOSS OF FOCK IMPLEMENTABILITY

The normalizable tail above is distinct from the ultraviolet singularity of an instantaneous perfect boundary removal. For the finite-box coefficient family

$$\beta^L_{mn} = (-1)^{n+(m-1)/2} \frac{2\sqrt{n}}{\pi\sqrt{m}\,(m+2n)}, \qquad m \text{ odd}, \tag{16}$$

the two sides of the shutter contribute equally. With a common mode cutoff K,

$$N_{\rm vac}(K) = \frac{8}{\pi^2} \sum_{\substack{1\le m\le K \\ m \text{ odd}}} \sum_{n=1}^{K} \frac{n}{m(m+2n)^2}. \tag{17}$$

Proposition 2 (Leading abrupt asymptotic)**.** *As* $K \to \infty$,

$$\boxed{N_{\rm vac}(K) = \frac{1}{2\pi^2}(\log K)^2 + O(\log K).} \tag{18}$$

Proof. Split the m sum into $m \le n$ and $m > n$. In the first region,

$$\frac{n}{m(m+2n)^2} = \frac{1}{4mn} + O(n^{-2}) \tag{19}$$

uniformly after summing over odd $m \le n$. Since

$$\sum_{\substack{m\le n \\ m \text{ odd}}} \frac{1}{m} = \frac{1}{2}\log n + O(1), \tag{20}$$

the leading contribution before the prefactor in Eq. (17) is $\frac{1}{16}(\log K)^2 + O(\log K)$. For $m > n$, the summand is bounded by n/m^3, whose double sum is $O(\log K)$. Multiplying by $8/\pi^2$ gives Eq. (18). □

The numerical fit gives 0.0508876 for the quadratic-log coefficient, within 0.448% of $1/(2\pi^2) = 0.0506606$ [Fig. 2]. More importantly,

$$\mathrm{Tr}(\beta\beta^\dagger) = \infty. \tag{21}$$

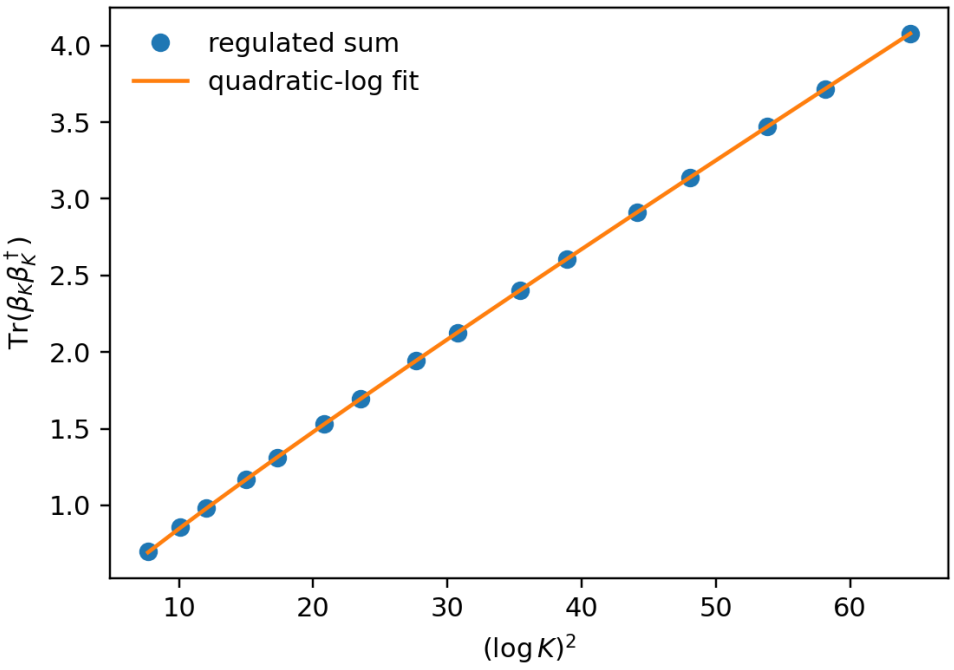

FIG. 2. Cutoff growth of the abrupt finite-box sum. The divergence of $\mathrm{Tr}(\beta_K\beta_K^\dagger)$, rather than a particular regulator convention, is the representation-theoretic criterion.

By the bosonic Shale criterion, the abrupt in/out transformation is not implemented by a unitary on the original out-Fock space[5]. Algebraic or enlarged-space implementations may still be considered[6], but the ideal limit is not a normalized Fock vector with a literal infinite-photon outcome.

IV. CONTINUUM THEOREM FOR A FINITE QUANTUM CONTROLLER

A. Quadratic-form model

Let $\mathfrak{h}$ be a separable one-particle Hilbert space and $\omega \geq 0$ a self-adjoint frequency operator. Set

$$H_f = \mathrm{d}\Gamma(\omega) \tag{22}$$

on $\mathcal{F}_s(\mathfrak{h})$. The controller Hilbert space $\mathcal{K}$ has fixed finite dimension, so every controller operator is bounded. Let $H_C = H_C^\dagger$, $Q_j \geq 0$, $\gamma_j \geq 0$, and

$$v_j \in \mathfrak{h}, \qquad \omega^{-1/2} v_j \in \mathfrak{h}, \qquad j = 1, \ldots, r. \tag{23}$$

With $\Phi(v) = a(v) + a^\dagger(v)$, define on

$$\mathcal{Q} = \mathcal{K} \otimes D(H_f^{1/2}) \tag{24}$$

the quadratic form

$$\begin{aligned} q[\Psi] &= \langle \Psi | (H_C \otimes \mathbb{I} + \mathbb{I} \otimes H_f) | \Psi \rangle \\ &\quad + \frac{1}{2} \sum_{j=1}^{r} \gamma_j \left\| (Q_j^{1/2} \otimes \Phi(v_j)) \Psi \right\|^2 . \end{aligned} \tag{25}$$

This covers positive controller-conditioned stiffness changes and finite stable quadratic kernels that admit a positive-square factorization. Indefinite kernels require an additional relative-bound or positivity hypothesis and are not silently included.

Let $v_{j,K}$ be finite-mode form factors and define

$$\eta_K = \max_j \left[\|v_{j,K} - v_j\| + \left\| \omega^{-1/2}(v_{j,K} - v_j) \right\| \right]. \tag{26}$$

Let q_K be Eq. (25) with v_j replaced by $v_{j,K}$.

Lemma 3 (Field estimate)**.** *For v satisfying Eq. (23) and $\Psi \in D(H_f^{1/2})$,*

$$\|\Phi(v)\Psi\| \le 2 \left\| \omega^{-1/2} v \right\| \left\| H_f^{1/2}\Psi \right\| + \|v\| \, \|\Psi\| . \tag{27}$$

Proof. The standard Fock estimates give $\|a(v)\Psi\| \le \left\|\omega^{-1/2}v\right\| \left\|H_f^{1/2}\Psi\right\|$ and $\left\|a^\dagger(v)\Psi\right\|^2 = \|a(v)\Psi\|^2 + \|v\|^2 \|\Psi\|^2$[9,12]. The triangle inequality yields Eq. (27). □

Theorem 4 (Finite-controller continuum limit)**.** *Assume Eq. (23) and $\eta_K \to 0$. Then:*

(i) q and every q_K are densely defined, closed, and lower bounded on the common domain $\mathcal{Q}$, and therefore define unique self-adjoint operators H and H_K.

(ii) For a sufficiently large common shift $c > 0$, there is a K-independent constant C such that

$$\left\| (H_K + c)^{-1} - (H + c)^{-1} \right\| \le C\eta_K \tag{28}$$

for all sufficiently large K.

(iii) For every $T < \infty$ and $\Psi \in \mathcal{H} = \mathcal{K} \otimes \mathcal{F}_s(\mathfrak{h})$,

$$\sup_{|t| \le T} \left\| e^{-\mathrm{i}tH_K}\Psi - e^{-\mathrm{i}tH}\Psi \right\| \to 0. \tag{29}$$

(iv) For every density operator ρ,

$$\sup_{|t| \le T} \left\| e^{-\mathrm{i}tH_K}\rho e^{\mathrm{i}tH_K} - e^{-\mathrm{i}tH}\rho e^{\mathrm{i}tH} \right\|_1 \to 0. \tag{30}$$

Consequently, both controller and field reduced density operators converge in trace norm, uniformly on compact time intervals.

Expectations of every bounded observable, including Weyl operators, converge uniformly on compact time intervals.

Proof. Equip $\mathcal{Q}$ with the free form norm $\|\Psi\|_0^2 = \left\|H_f^{1/2}\Psi\right\|^2 + \|\Psi\|^2$. Lemma 3 makes each map $Q_j^{1/2} \otimes \Phi(v_j) : \mathcal{Q} \to \mathcal{H}$ bounded. The positive interaction forms are therefore continuous in $\|\cdot\|_0$. Adding finitely many of them to the free closed form and the bounded controller term preserves closedness and lower boundedness, proving (i).

For the associated sesquilinear interaction forms,

$$|(q_K - q)[\Psi, \Xi]| \le C_0 \eta_K \|\Psi\|_0 \|\Xi\|_0 , \tag{31}$$

where uniform boundedness of $v_{j,K}$ in both norms follows from $\eta_K \to 0$. After a common shift, the form norm of $q + c$ is equivalent to $\|\cdot\|_0$. By the Riesz representation theorem there is a bounded self-adjoint D_K on the shifted form Hilbert space with $\|D_K\| \le C_1\eta_K$ such that

$$(q_K + c)[\Psi, \Xi] = (q + c)[\Psi, \Xi] + \langle (H + c)^{1/2}\Psi, D_K (H + c)^{1/2}\Xi \rangle. \tag{32}$$

Hence, in the quadratic-form sense,

$$(H_K + c)^{-1} = (H + c)^{-1/2}(\mathbb{I} + D_K)^{-1}(H + c)^{-1/2}. \tag{33}$$

For $\|D_K\| < 1/2$, the resolvent identity for $(\mathbb{I} + D_K)^{-1}$ gives Eq. (28). This is the standard common-domain form-norm argument[10].

Norm-resolvent convergence implies strong-resolvent convergence. The self-adjoint Trotter–Kato theorem then gives Eq. (29), uniformly for t in compact intervals[13]. For a rank-one operator $|x\rangle\langle y|$, the trace-norm difference of the two evolutions is bounded by

$$\|(U_K - U)x\| \, \|y\| + \|x\| \, \|(U_K - U)y\| \, . \tag{34}$$

Finite-rank approximation extends this uniformly to any trace-class ρ, proving Eq. (30). Partial trace is contractive in trace norm. □

Corollary 5 (Finite interacting-mode regulators)**.** *Let P_K be finite-rank projections with $v_{j,K} = P_K v_j$ and $\eta_K \to 0$. Then Theorem 4 applies. If P_K reduces ω, Fock factorization separates the complement into a free spectator field. For an initial spectator vacuum, the interacting evolution is exactly the corresponding finite-mode Galerkin calculation embedded in the full Fock space.*

Corollary 6 (Unbounded number observables)**.** *If, in addition to Theorem 4, the evolved states obey a uniform bound*

$$\sup_{K, |t| \le T} \mathrm{Tr}[\rho_K(t)(N + \mathbb{I})^{1+\delta}] < \infty \tag{35}$$

for some $\delta > 0$, then $\mathrm{Tr}[\rho_K(t)N]$ converges uniformly on compact time intervals. A mass gap $\omega \ge m\mathbb{I}$, together with a suitable uniform energy moment, is one sufficient route to Eq. (35)*.*

Remark 7 (Singular limits). Theorem 4 does not cover a delta-like perfect boundary whose form factors fail Eq. (26), nor an infinitely abrupt switch whose Bogoliubov map violates the Shale condition. Those are singular limits, not counterexamples to the smooth-coupling theorem.

V. NUMERICAL AUDITS

We tested Theorem 4 on

$$H_K = \mathrm{d}\Gamma(\omega_K) + \frac{\Omega}{2}X + \frac{\gamma}{2}Q\Phi(v_K)^2, \tag{36}$$

with a two-level controller, $Q = |0\rangle\langle 0|$, $\gamma = 0.22$, $\Omega = 1.15$, and

$$\omega_k = \sqrt{0.8^2 + (\pi k/5)^2}, \qquad v_k \propto \frac{e^{-(k/4)^2}}{\sqrt{\omega_k}}. \tag{37}$$

The initial state is $|0\rangle_C \otimes |0\rangle_F$ and the evolution time is $T = 3$. A global number ceiling is used only to audit the finite numerical calculation; it is absent from Theorem 4. Sparse exponential action is evaluated by a Krylov method[14].

At $n_{\max} = 6$, the controller trace distance to the $K = 14$ result decreases as shown in Table I. The empirical fit is proportional to $\eta_K^{1.915}$; this exponent is model dependent and is not asserted by the theorem, which guarantees convergence and the general linear upper bound in Eq. (28).

At $K = 8$, raising the global number ceiling from 6 to 8 changes $\langle N \rangle$ by 2.17×10^{-6} and the controller state by trace distance 2.63×10^{-6}; the $n_{\max} = 8$ top-shell probability is 6.62×10^{-9}. Figure 3 shows both regulator checks.

The Gaussian certificate has symplectic residual 4.29×10^{-13}, while the abrupt coefficient fit agrees with Proposition 2 to 0.448%. These calculations test the constructions used in the applications; they are not substitutes for the analytic proofs.

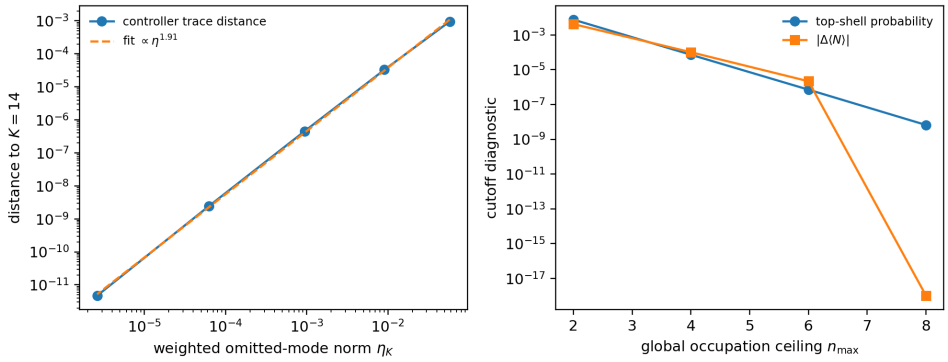

FIG. 3. Numerical continuum audit. (a) Controller trace distance versus the weighted omitted-mode norm η_K, using $K = 14$ as the finite reference. (b) Independent occupation-ceiling diagnostics at $K = 8$. The theorem itself has neither a finite mode count nor a Fock occupation cutoff.

VI. SCOPE AND IMPLICATIONS

The results establish three distinct facts.

First, a normalizable photon-added squeezed state can have strict support in every retained number sector in the *fixed physical outgoing basis*. The conditions $Z_{AB} \neq 0$ and $f_B \neq 0$ are easy to test and do not depend on diagonalizing the reduced state. Local same-side squeezing cannot remove the tail because the proof isolates a unique highest-degree cross-pair contribution.

Second, the ideal abrupt perfect-boundary limit is qualitatively different. Its anomalous block is not Hilbert–Schmidt, so the in/out vacua belong to inequivalent Fock representations. Divergent mean number and unbounded support should therefore not be described as the same phenomenon.

Third, a finite quantum controller can remain fully quantum while the interacting field-mode regulator is removed. The controller does not have to become classical or infinite dimensional before the continuum limit. Under the weighted form-factor hypothesis, convergence propagates from quadratic forms to resolvents, unitary dynamics, full density operators, and reduced states.

The theorem is deliberately not a renormalization theorem for arbitrary point interactions. It treats smooth smeared positive quadratic couplings, or stable finite kernels that admit such a factorization. It does not establish a continuum Hamiltonian for an instantaneous perfect mirror, an ultraviolet-singular delta shutter, or a generic indefinite quadratic kernel without a separate stability analysis. Nor does it identify any particular microscopic substrate as the fundamental description of electromagnetism.

TABLE I. Finite-mode audit at $n_{\max} = 6$.

K	$D(\rho_C^{(K)}, \rho_C^{(14)})$	$\lvert\Delta\langle N\rangle\rvert$
4	9.6401×10^{-4}	3.7212×10^{-5}
6	3.3045×10^{-5}	6.1888×10^{-7}
8	4.5129×10^{-7}	7.4653×10^{-9}
10	2.4413×10^{-9}	2.6648×10^{-11}
12	4.6970×10^{-12}	4.8964×10^{-14}

VII. CONCLUSION

A photon-added Gaussian state with nonzero cross-sector squeezing and a nonzero discarded-sector photon component has a retained number distribution with

$$P_A(n) > 0 \qquad \forall n \in \mathbb{N}_0, \tag{38}$$

with no occupation cutoff. The result holds directly in the specified outgoing mode basis. The abrupt perfect-boundary limit must instead be described as non-Fock because $\mathrm{Tr}(\beta\beta^\dagger)$ diverges.

For smooth autonomous couplings, a fixed finite-dimensional quantum controller has a controlled field continuum limit: Hamiltonians converge in norm resolvent, dynamics converge strongly on compact time intervals, and full and reduced states converge in trace norm. Together, these statements supply the mathematical closure needed to separate a normalizable "zero-to-arbitrarily-large" number tail from a singular ideal quench, while retaining a genuinely quantum actuator.

ACKNOWLEDGMENTS

The numerical data and figure-generation scripts accompanying this article are archived in the reproducibility package. The author used computational and editorial assistance during development; all claims, derivations, and numerical thresholds were independently checked against the supplied code and data.

DATA AVAILABILITY

All scripts, numerical outputs, figure sources, and verification checks required to reproduce the reported audits are included in the accompanying archive. A permanent repository identifier will be inserted before journal submission.

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