Escape Routes Beyond the Compact Three-Dimensional TT Pairing Obstruction

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Abstract

A prior compact CSS/clock audit found that the local logical symplectic form of the tested three-dimensional tensor code vanishes as k^3. In that class, bounded-range analytic positive Hamiltonians consequently give $H_{pp} = \mathcal{O}(k^2)$, $H_{pq} = \mathcal{O}(k^3)$, $H_{qq} = \mathcal{O}(k^4)$ and generically $\omega \sim k^3$, rather than a relativistic helicity-2 branch. This note tests five proposed escape routes: a noncompact auxiliary Schur parent, gapless auxiliaries, nonlocal inverse-derivative variables, dressed rather than exact constraints, and a different square-root register algebra. The noncompact constrained parent yields two exact $z = 1$ TT branches with a finite auxiliary gap. Positive analytic gapless auxiliary blocks cannot lower the effective k^4 stiffness without becoming indefinite. Inverse-derivative canonicalization yields $z = 1$ but has algebraic real-space tails and a logarithmically divergent volume norm. A gapped dressed projector is exponentially local, yet dressing preserves the starting dynamical exponent and does not forbid an allowed mass. Finally, square-root ladders converge to the noncompact oscillator only in a joint limit in which local dimension grows approximately as the inverse soft frequency. No tested route supplies a fixed finite-dimensional, local, exact-constraint, generic three-dimensional $z = 1$ tensor phase.

1 Question and scope

The compact obstruction was not a no-go theorem for every microscopic tensor model. It applied to translation-invariant finite-range clock/Pauli-centralizer Hamiltonians built from a specific exact CSS constraint algebra, expanded analytically around a stable minimum. The present calculation deliberately probes routes outside that class.

The target remains a local theory with two helicity-2 modes,

$$\omega_{\pm 2}(\boldsymbol{k}) = c_T|\boldsymbol{k}| + \mathcal{O}(k^3), \tag{1}$$

with a nonclosing separation from constraint defects and no symmetry-allowed TT mass. A successful regulator must also state whether its local Hilbert-space dimension remains fixed or grows in the continuum limit.

Table 1 summarizes the outcome. Each verdict is restricted to the model class actually computed.

2 Noncompact auxiliary Schur parent

Write the six orthonormal components of the symmetric tensor as h. With lattice derivative symbol d_i, define the divergence Vh, the trace-gradient Gh, and

$$L = \begin{pmatrix} 2V - G \\ G \end{pmatrix}. \tag{2}$$

Route	Result	Precise scope
Noncompact auxiliary Schur parent	Positive	Exact constrained quadratic theory; two TT branches with $z = 1$ and a finite auxiliary gap. The scalar constraint is load-bearing.
Gapless auxiliary fields	Negative in tested class	For positive analytic second-order blocks, positivity prevents a Schur correction of order k^2.
Inverse-derivative variables	Positive but nonlocal	Canonical normalization produces $z = 1$ only with algebraic tails and no volume-uniform L^2 norm.
Dressed constraints	Stability only	A defect-band projector remains exponentially local, but dressing neither changes z nor forbids a mass.
Square-root register algebra	Double-scaling only	The noncompact oscillator is recovered when local dimension grows approximately as $1/\omega$; fixed dimension crosses to quadratic infrared behavior.

Table 1: Summary of the five escape-route tests.

The linearized-Einstein stiffness can be factored as

$$K_E(d) = |d|^2 I_6 - \frac{1}{2} L(d)^T L(d). \tag{3}$$

Introducing six auxiliary coordinates y gives the local parent

$$M(d) = \begin{pmatrix} |d|^2 I_6 & L(d)^T \\ L(d) & 2I_6 \end{pmatrix}. \tag{4}$$

Its Schur complement is exactly K_E. Under

$$\delta h = D(d)\xi, \qquad \delta y = -\frac{1}{2} L(d) D(d) \xi, \tag{5}$$

we find $M D_{\text{ext}} = 0$ to a maximum numerical residual of 8.96×10^{-16}. The Schur identity is satisfied to 6.66×10^{-16}.

After quotienting the gauge directions and imposing the scalar constraint, the lowest two stiffness eigenvalues are $|d|^2$. Three generic directions give fitted exponents

$$z = 0.9997813, \quad 0.9998602, \quad 0.9998946, \tag{6}$$

with TT weights equal to one within floating-point precision. The first non-TT frequency in the canonical scan is $\sqrt{2}$.

This is a genuine positive result, but two qualifications are essential. First, the coordinates are noncompact. Second, the scalar constraint is not optional: removing it exposes one negative scalar stiffness whose magnitude scales as $k^{1.9981}$. Equation (4) is therefore an exact constrained parent, not an unconstrained positive oscillator array.

3 Can gapless auxiliaries lower the compact order?

Consider the scalar order-counting block

$$H(k) = \begin{pmatrix} bk^4 & ck^\gamma \\ ck^\gamma & ak^{2s} \end{pmatrix}, \qquad a, b > 0. \tag{7}$$

Here the first entry represents the compact physical curvature stiffness, while the second is a possibly gapless auxiliary. Positivity requires

$$\det H \geq 0 \quad \Longrightarrow \quad 2\gamma \geq 4 + 2s, \qquad \gamma \geq s + 2. \tag{8}$$

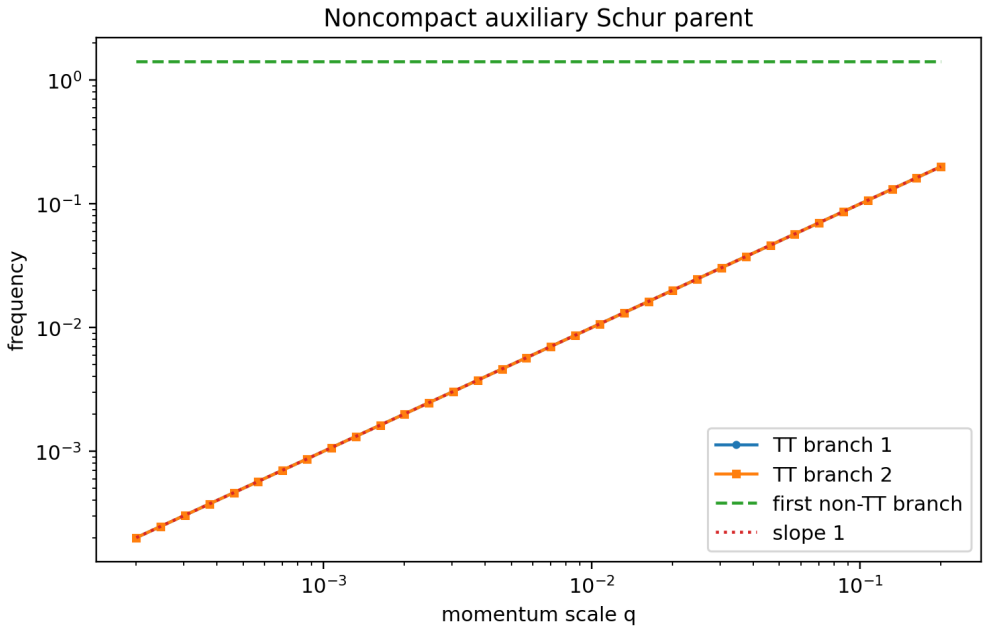

Figure 1: Two degenerate TT branches of the noncompact auxiliary parent. The auxiliary sector remains gapped while the physical branches are linear.

The Schur correction then obeys

$$\frac{C^2}{A} = \mathcal{O}\big(k^{2\gamma-2s}\big) = \mathcal{O}(k^4) \quad \text{or higher.} \tag{9}$$

To generate a k^2 term one would need $\gamma = s+1$, which violates positivity at sufficiently small momentum.

The numerical scan used $s = 0, 1, 2, 3$. Every stable choice with $\gamma = s + 2$ retained an effective k^4 stiffness and, with the compact k^2 kinetic block, $z = 3$. Every attempted $\gamma = s + 1$ block developed a negative eigenvalue.

This is a no-go only for positive analytic second-order Schur blocks. It does not cover singular first-order constrained systems, indefinite Lagrange-multiplier formulations, or an auxiliary field that already carries the desired helicity-2 mode.

4 Nonlocal inverse-derivative variables

For the compact logical generators, the nonzero singular values of the five-by-five symplectic pairing satisfy

$$\sigma_{1,2}[\Omega(\boldsymbol{k})] \propto |k|^3. \tag{10}$$

Across 80 random directions, both fitted powers were exactly three at displayed precision. A symmetric canonical normalization therefore contains

$$\Omega^{-1/2} \sim |k|^{-3/2}. \tag{11}$$

In these nonlocal variables one may define an ordinary canonical Hamiltonian with $\omega = |d|$; the measured exponent is $z = 0.9998621$.

The cost is real-space nonlocality. A scalar representative of the least-singular normalization has a radial kernel

$$K(r) \sim r^{-3/2}. \tag{12}$$

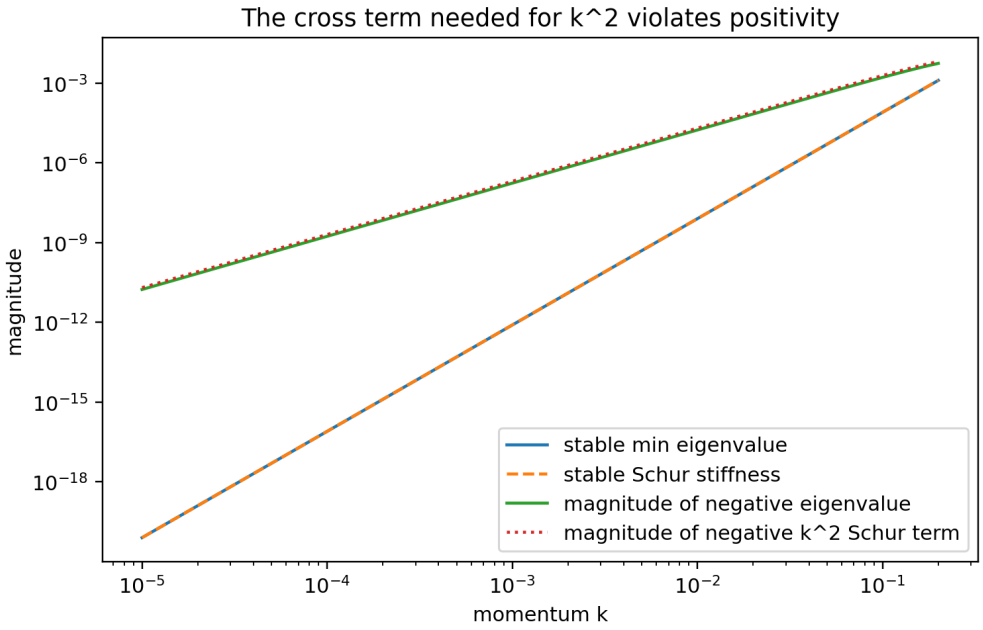

Figure 2: For a representative gapless auxiliary, the stable cross term leaves a k^4 Schur stiffness. The lower-order cross term required for k^2 gives a negative eigenvalue.

The numerical fit gives -1.49650. Moreover, Parseval's identity gives a squared norm growing logarithmically with volume,

$$\frac{1}{V}\sum_{\boldsymbol{k}\neq 0}|d(\boldsymbol{k})|^{-3} = 0.0506530\log L + 0.0268363, \tag{13}$$

with $R^2 = 0.99999998$.

Thus inverse derivatives do evade the compact order obstruction, but only by abandoning bounded-range locality. The matrix-valued orientation dependence changes angular structure, not the singular radial order.

5 Dressed rather than exact constraints

A separate six-band local model was used to isolate what quasi-local dressing can and cannot accomplish. Two low bands have either $\omega \sim k$ or $\omega \sim k^3$ and are separated from four defect bands by a finite gap. A finite-range invertible mixing is applied, and the low-band Riesz projector is then reconstructed from the perturbed spectrum rather than supplied by hand.

For $L = 24, 32, 48$, the projector profiles are volume independent before the roundoff floor. The fitted localization length is

$$\xi_P = 0.242465 \text{ lattice sites}, \tag{14}$$

while the minimum low/defect separation is 4.00380. The commutator residual $\|[H, P]\|$ is 1.25×10^{-14}.

However, dressing preserves the starting critical exponent: the $z = 1$ parent remains $z = 0.999964$, while the $z = 3$ parent remains $z = 2.99853$. An allowed onsite mass term opens a zero-momentum gap 0.119681 without destroying projector locality. Therefore quasi-local dressing can stabilize a sector already present in the parent, consistent with quasi-adiabatic ideas for gapped band separation [2]; it cannot by itself create a relativistic tensor mode or prohibit a mass.

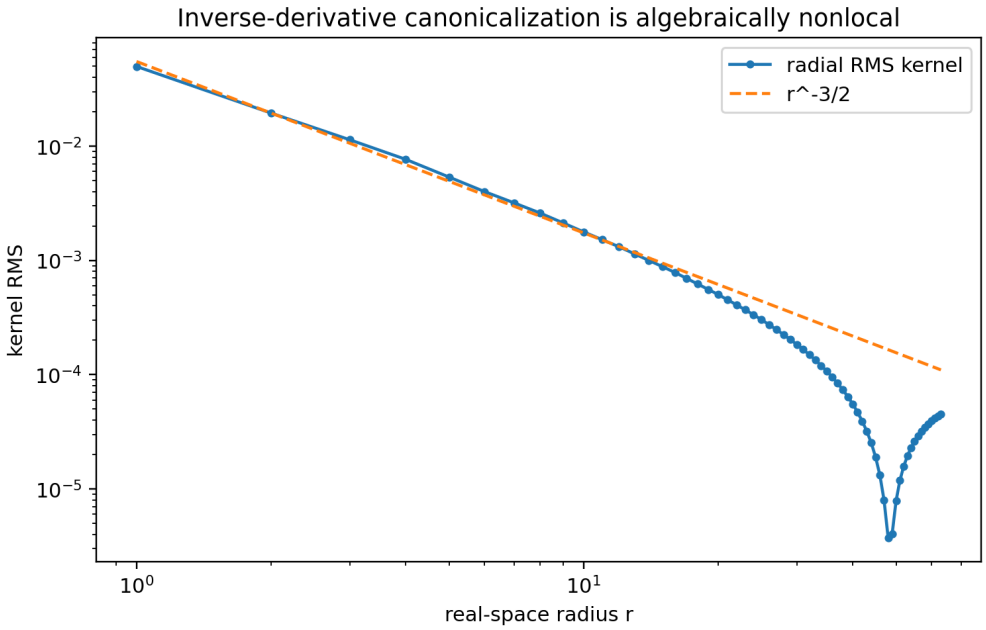

Figure 3: Radial real-space kernel of the symmetric inverse-derivative normalization. The controlled short-distance fit approaches $r^{-3/2}$; finite-volume zero-mode subtraction distorts the far tail.

The object reconstructed here is a dressed band projector, not a complete set of local non-Abelian Gauss generators. This route is therefore a mechanism screen rather than tensor closure.

6 A different register algebra: square-root ladders

The exact compact centralizer algebra can be replaced by truncated Heisenberg registers with

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{15}$$

For a single oscillator of target frequency ω, the finite matrix

$$H_d(\omega) = \frac{1}{2}p_d^2 + \frac{1}{2}\omega^2 q_d^2 \tag{16}$$

was diagonalized as the local dimension d increased.

To keep the lowest gap within one percent of ω, the required dimension obeys

$$d_{1\%} \simeq 5.0414\,\omega^{-0.95849}, \qquad R^2 = 0.99910, \tag{17}$$

with mean $d\omega = 4.5863$ over the tested range. The scaling approaches $d \propto 1/\omega$.

At fixed d, the deepest infrared instead crosses to

$$\Delta_d(\omega) \propto \omega^2. \tag{18}$$

The fitted powers range from 2.00013 at $d = 8$ to 1.98447 at $d = 32$ over the controlled low-frequency window.

For the lowest mode in a box, $\omega_{\min} \sim L^{-1}$, so fixed relative accuracy requires $d \sim L$. The square-root register is a valid double-scaling regulator of the noncompact parent; it is not a fixed finite-dimensional thermodynamic realization. Unit-amplitude ladders fail even this low-energy oscillator test.

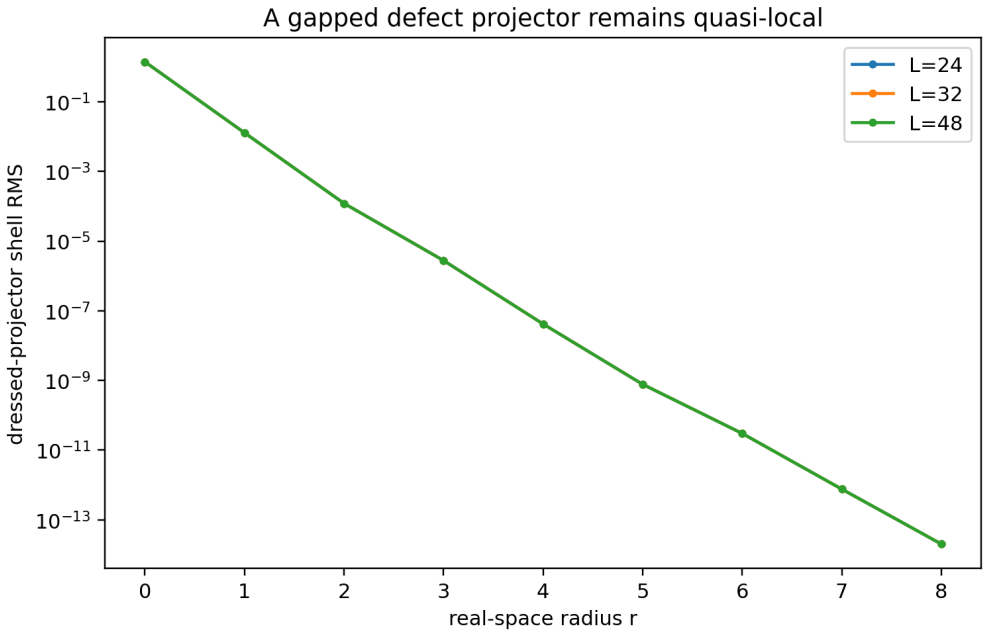

Figure 4: The defect-band projector remains exponentially localized and volume uniform under a finite-range dressing.

7 Combined verdict and next target

The five routes sharply separate three notions that had previously been conflated:

1. **A local constrained quadratic mechanism.** The noncompact auxiliary Schur parent supplies this and gives exactly two $z = 1$ TT branches.
2. **A fixed finite-dimensional compact phase.** None of the tested routes establishes this in generic three dimensions.
3. **A regulator sequence approaching the continuum.** Square-root ladders provide one, but their local dimension must grow with volume.

The gapless-auxiliary and dressed-constraint routes do not independently solve the dynamical-order problem. Nonlocal canonicalization solves it only by sacrificing locality. The best-supported constructive direction is therefore to treat the noncompact auxiliary parent as the continuum target and ask whether a controlled family of finite-register models converges to it, possibly using a quantum-group or other non-CSS regulator whose local dimension is allowed to increase. A fixed-dimensional alternative remains logically possible, but it must use a microscopic algebra outside the positive compact centralizer class and demonstrate its phase by a genuine many-body calculation.

This conclusion is compatible with earlier spin-model work in which a controlled model produced helicity-2 modes with k^3 dispersion, whereas a proposed linear-dispersion construction required a less controlled strong-coupling argument [1]. The present result should therefore be read as a route classification, not as a general no-go theorem.

Reproducibility

The accompanying archive contains the standalone Python calculation, aggregate JSON data, figures, and checksums. The full run completes in approximately eleven seconds on the supplied environment. All numerical statements in this note are generated from that one script.

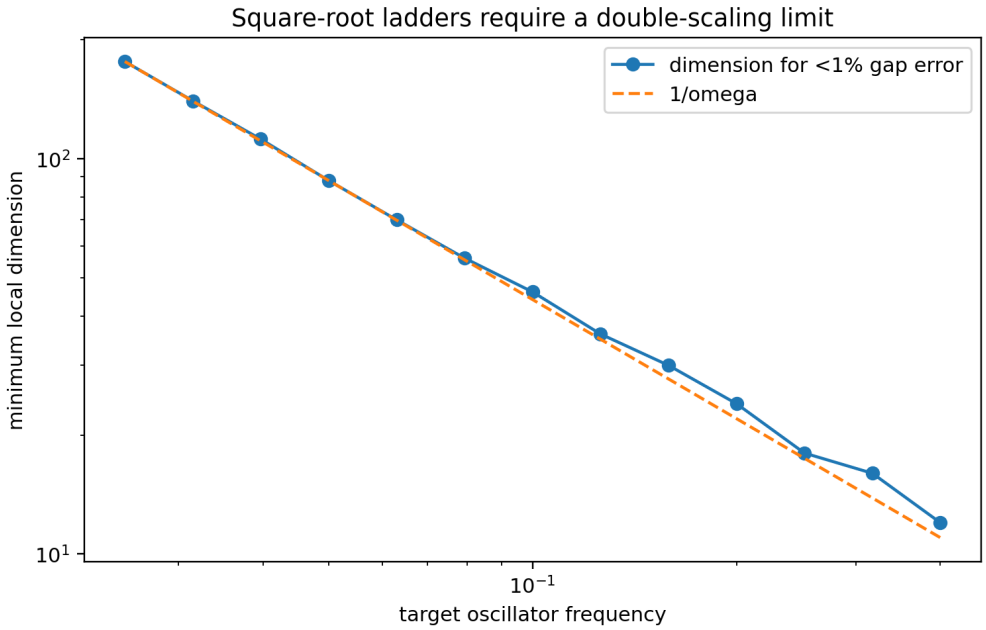

Figure 5: Local dimension required for one-percent reproduction of the noncompact oscillator gap.

References

[1] Z.-C. Gu and X.-G. Wen, "Emergence of helicity ±2 modes (gravitons) from qubit models," *Nucl. Phys. B* **863**, 90–129 (2012), arXiv:0907.1203.

[2] M. B. Hastings and X.-G. Wen, "Quasi-adiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance," *Phys. Rev. B* **72**, 045141 (2005), arXiv:cond-mat/0503554.

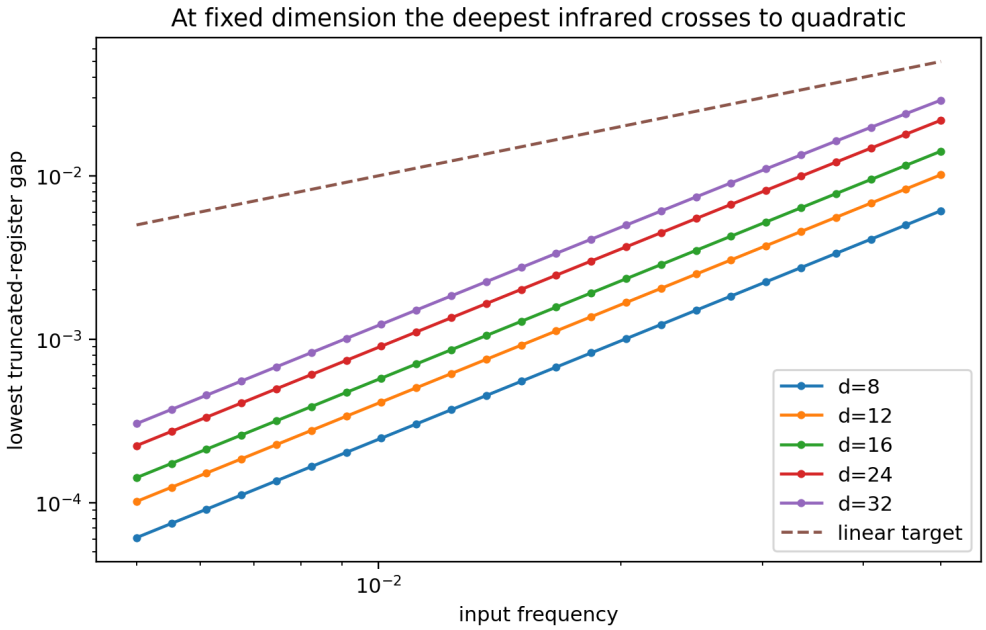

Figure 6: At fixed local dimension, the softest oscillator gap becomes quadratic in the input frequency. Increasing d pushes the crossover to lower frequency but does not remove it at fixed d.