An Axis-Span Obstruction to Selective Gapping in Regular Harmonic Symmetric-Tensor Models

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Abstract

Can finite quadratic penalties leave only two gapless tensor polarizations in a common six-component symmetric-tensor coordinate space? We give a short obstruction for positive, regular harmonic models. The transverse-traceless subspaces for the three coordinate-axis propagation directions together span all five traceless components. Consequently, a positive-semidefinite zero-momentum stiffness preserving these limits annihilates the entire traceless subspace and can penalize only the trace. With regular positive inertia and ordinary second-order dynamics, at least five normal-mode frequencies therefore approach zero. Finite derivative penalties and regular static elimination of uniformly stiff auxiliaries do not evade the argument. The result requires no rotational invariance. We distinguish this coefficient-independent certificate from earlier penalty-failure arguments of Gu and Wen. The claim is limited to the stated harmonic class; it does not exclude compact non-Gaussian phases, gauge quotients, or more general dynamics.

1 Question and scope

A symmetric tensor in three spatial dimensions has six independent components. For each nonzero propagation direction, its transverse-traceless (TT) subspace has dimension two. A proposed harmonic realization may try to retain those two soft polarizations while making the other components massive through finite local penalties. The difficulty is that the desired two-dimensional subspace changes with direction.

The broad failure of some finite penalties to enforce low-energy constraints is already explicit in Gu and Wen [1]. Our purpose is narrower: to state and prove a small linear-algebra certificate for a specified regular harmonic class. This manuscript develops the proof of an August 30, 2026 retrospective research certificate [2]; it reports no new many-body calculation or numerical phase identification.

Let

$$u = (S_{xx}, S_{yy}, S_{zz}, S_{xy}, S_{xz}, S_{yz})^T, \qquad t = (1, 1, 1, 0, 0, 0)^T. \tag{1}$$

We use the ordinary Euclidean inner product on these six coordinates, not the Frobenius normalization that counts off-diagonal entries twice. This convention does not affect the span argument. The traceless subspace is

$$V_0 = \{u : t^T u = 0\} \simeq \mathrm{Sym}_0^2(\mathbb{R}^3). \tag{2}$$

For a unit vector n, write $T_n = \{S = S^T : \operatorname{tr} S = 0,\ Sn = 0\}$, represented in the coordinates (1).

Consider a translation-invariant harmonic model with coordinate stiffness $K(\boldsymbol{k})$, Hermitian and positive semidefinite near $\boldsymbol{k} = 0$, continuous there, and real symmetric at zero. Real finite-range

quadratic interactions give an analytic Fourier symbol and satisfy these regularity conditions when the harmonic energy is nonnegative. Positivity is an assumption, not a consequence of locality.

We say that the model preserves the *axis-TT zero-energy limits* if

$$\lim_{\epsilon\downarrow 0} v^\dagger K(\epsilon \boldsymbol{e}_a) v = 0 \quad \text{for every } v \in T_{\boldsymbol{e}_a}, \qquad a = x, y, z. \tag{3}$$

This is the relevant coordinate-space requirement when two gapless eigenmodes approach the TT plane on each axis with regular inertia. Only three axes are needed; requiring all directions includes this condition. An arbitrary pair of soft modes with no such TT limits does not satisfy the hypothesis.

2 The axis-span theorem

Lemma 1 (Six witnesses)**.** *The sum $T_{\boldsymbol{e}_x} + T_{\boldsymbol{e}_y} + T_{\boldsymbol{e}_z}$ is V_0 and has dimension five.*

Proof. In the component order (1), choose

$$\begin{array}{lll} & v_{a,+} & v_{a,\times} \\ a = x & (0,1,-1,0,0,0)^T & (0,0,0,0,0,1)^T \\ a = y & (1,0,-1,0,0,0)^T & (0,0,0,0,1,0)^T \\ a = z & (1,-1,0,0,0,0)^T & (0,0,0,1,0,0)^T. \end{array} \tag{4}$$

Each pair is traceless, transverse to its indicated axis, and spans that axis's TT plane. The three cross vectors span the three independent off-diagonal components. The first two plus vectors are independent and span the diagonal traceless components. The remaining plus vector satisfies $v_{z,+} = v_{y,+} - v_{x,+}$. Thus the six witnesses span precisely the five-dimensional space V_0. □

Lemma 2 (Zero quadratic energy)**.** *If A is Hermitian positive semidefinite and $v^\dagger A v = 0$, then $Av = 0$.*

Proof. $v^\dagger A v = \|A^{1/2} v\|^2$, so $A^{1/2} v = 0$, and hence $Av = 0$. □

Theorem 3 (Only the trace can acquire a zero-momentum mass)**.** *Under the stiffness and axis-TT assumptions of Section 1,*

$$V_0 \subseteq \ker K(0), \qquad K(0) = \mu\, t t^T, \qquad \mu \geq 0. \tag{5}$$

In particular, $\dim \ker K(0) \geq 5$ *and* $\operatorname{rank} K(0) \leq 1$.

Proof. Continuity and (3) give $v_{a,\sigma}^T K(0) v_{a,\sigma} = 0$ for all six witnesses. Lemma 2 puts them in $\ker K(0)$, and Lemma 1 puts all of V_0 there. Since $K(0)$ is symmetric, its range lies in $V_0^\perp = \operatorname{span}\{t\}$. A symmetric matrix with that range and kernel has the form $\mu t t^T$; positivity requires $\mu \geq 0$. □

No isotropic ansatz for $K(0)$ was used. In particular, arbitrary anisotropic constant coefficients do not evade the theorem if the same axis-TT limits must survive. In the chosen coordinates the remaining energy is

$$\tfrac{1}{2} u^T K(0) u = \tfrac{\mu}{2} (\operatorname{tr} S)^2. \tag{6}$$

The theorem permits a trace mass; it does not require $\mu > 0$.

3 Dynamical consequences and regular auxiliaries

3.1 What the kernel says about frequencies

To convert stiffness zeros into frequency zeros, assume ordinary second-order equations

$$M(\boldsymbol{k})\ddot{u} + K(\boldsymbol{k})u = 0, \qquad 0 < m_- I \le M(\boldsymbol{k}) \le m_+ I \tag{7}$$

near zero momentum, with M continuous and Hermitian. The constants $m_\pm$ are finite and independent of $\boldsymbol{k}$. There are no coordinate–momentum cross terms, Berry terms, or gyroscopic terms.

Corollary 4 (At least five zero-frequency limits)**.** *Under* (7) *and Theorem 3, the five lowest normal-mode frequencies tend to zero as* $\boldsymbol{k} \to 0$*, along every direction.*

Proof. The squared frequencies are the ordered eigenvalues of the continuous positive-semidefinite matrix

$$D(\boldsymbol{k}) = M(\boldsymbol{k})^{-1/2} K(\boldsymbol{k}) M(\boldsymbol{k})^{-1/2}. \tag{8}$$

Invertible congruence preserves nullity, so $D(0)$ has at least five zero eigenvalues. Continuity of the ordered eigenvalues of a Hermitian matrix gives $\omega_j(\boldsymbol{k})^2 \to 0$ for $j = 1, \dots, 5$. Taking nonnegative square roots proves the claim. □

This counts zero-frequency limits, including possibly flat zero modes. It establishes neither propagating excitations nor a particular exponent, residue, or source visibility. With $\mu = 0$ all six frequencies tend to zero. In isotropic examples the three excess traceless channels may be classified as two vector channels and one scalar channel relative to a fixed direction; that classification is not needed for the proof and can mix in anisotropic models.

3.2 Finite derivative penalties

A finite quadratic derivative penalty has a Fourier contribution of the form

$$\Delta K(\boldsymbol{k}) = \sum_\alpha c_\alpha B_\alpha(\boldsymbol{k})^\dagger W_\alpha(\boldsymbol{k}) B_\alpha(\boldsymbol{k}), \qquad B_\alpha(0) = 0, \tag{9}$$

where the symbols B_α are continuous, the coefficients c_α are fixed and finite, and the matrices W_α are bounded near zero. Thus $\Delta K(0) = 0$. Provided the full stiffness remains positive semidefinite and the target TT limits are preserved, such terms can alter dispersion coefficients and mixing but cannot remove the five-dimensional zero-momentum kernel. Constant penalties are instead restricted by (5). Taking an infinite penalty before the infrared limit, or projecting onto an exact constraint sector, is a different operation and is not covered.

3.3 Static elimination of uniformly stiff auxiliaries

Let a finite set of auxiliary coordinates y give the full stiffness

$$\mathcal{K}(\boldsymbol{k}) = \begin{pmatrix} A(\boldsymbol{k}) & B(\boldsymbol{k}) \\ B(\boldsymbol{k})^\dagger & C(\boldsymbol{k}) \end{pmatrix} \succeq 0, \qquad C(\boldsymbol{k}) \succeq cI, \quad c > 0, \tag{10}$$

with analytic blocks near zero. Minimization over y gives

$$K_{\text{eff}} = A - BC^{-1}B^\dagger \succeq 0, \qquad y_* = -C^{-1}B^\dagger u. \tag{11}$$

Both statements follow by completing the square:

$$\begin{pmatrix} u \\ y \end{pmatrix}^{\dagger} \mathcal{K} \begin{pmatrix} u \\ y \end{pmatrix} = u^{\dagger} K_{\text{eff}} u + (y + C^{-1} B^{\dagger} u)^{\dagger} C (y + C^{-1} B^{\dagger} u). \tag{12}$$

The inverse of C is analytic near zero. If K_{eff} preserves the axis-TT limits, Theorem 3 applies to it. At zero, the map $u \mapsto (u, -C^{-1}B^{\dagger}u)$ identifies $\ker K_{\text{eff}}$ with $\ker \mathcal{K}$. Consequently the full system also has at least five zero-frequency limits if its full inertia satisfies (7).

This is a static statement. It does not replace dynamical elimination by a frequency-independent approximation. Nor does finite range of the full blocks imply finite range of their Schur complement: the inverse generally introduces tails. A singular or gapless auxiliary block, or a singular inertia, invalidates the corresponding hypotheses.

4 Comparison with Gu and Wen and the contribution claimed

Gu and Wen's vector-field examples in Sections III.2–III.3, Eqs. (8)–(10), retain an unwanted longitudinal soft mode after a finite divergence penalty [1]. Their tensor discussion in Section V.1 also identifies continuous constraint spectra as a problem for finite penalties. Sections III.4 and V.2 then introduce compactification and discrete constraint spectra; the tensor construction uses periodic operators in Eqs. (39)–(40). These observations predate this work. The claim that ordinary penalties can fail is therefore not new.

The present statement has a different mathematical target: the zero-momentum *coordinate stiffness* of a common six-component harmonic sector. Gu and Wen's vector example includes a penalty in the momentum sector. We do not claim that our assumptions encompass all their models, or that the two arguments are interchangeable. Their compact construction is not refuted by this theorem.

The contribution offered here is the following explicit formulation:

1. Six TT witnesses on only three axes force a five-dimensional common kernel, without rotational invariance.
2. Positivity determines the entire admissible constant mass family, $K(0) = \mu t t^{T}$, rather than testing selected coefficients.
3. Regular inertia gives a precise frequency corollary, while the Schur-complement argument specifies which auxiliary repairs remain subject to the same obstruction.

This is an elementary synthesis of tensor geometry and positive linear algebra. The claimed contribution is its explicit certificate and scoped application, not a new general principle of constraint physics. No first-in-literature priority claim is made: an equivalent formulation elsewhere in the literature has not been excluded.

5 Limitations and conclusion

The theorem applies to the specified common coordinate sector and to nonsingular analytic changes of those coordinates, with the TT subspaces transformed consistently. It does not apply to exact or emergent gauge quotients, genuinely non-Gaussian phases lacking this regular harmonic description, nonanalytic or nonlocal TT projectors, singular auxiliary reductions, first-order or gyroscopic dynamics, or Floquet/QCA evolution. Choosing two fixed tensor components while abandoning the directional TT target also changes the question.

The obstruction concerns an infrared family of Fourier symbols. A gap at the smallest nonzero momentum in one finite box is not an infrared gap. The proof does not construct a bounded microscopic Hamiltonian, establish a many-body phase, or make a statement about gravity in nature.

Within its assumptions, the conclusion is decisive: preserving the required two TT limits on three axes preserves all five traceless zero-energy directions. Finite regular quadratic retuning cannot selectively gap the three unwanted directions. A successful mechanism must change at least one of those assumptions.

Provenance and drafting note. The underlying certificate [2] is retrospective, dated August 30, 2026, and records rank five for the six witnesses and rank twenty for their action on a general symmetric 6×6 mass matrix with twenty-one coefficients. The proof above is self-contained and does not require rerunning that calculation. This October 24 draft was prepared with AI assistance and requires author review before submission; no new scientific decision run was performed.

References

[1] Z.-C. Gu and X.-G. Wen, *Emergence of helicity ±2 modes (gravitons) from qubit models*, Nuclear Physics B **863**, 90–129 (2012). doi:10.1016/j.nuclphysb.2012.05.010. arXiv:0907.1203v3 (2010; first version 2009). Section and equation numbers cited here refer to the arXiv v3 text.

[2] J. J. Farrow research archive, *Native Tensor Soft-Penalty Obstruction Results*, unpublished internal research record, August 30, 2026; repository `qca_test_qatnu`, branch `codex/finite-spin-dsf`, scope revision `1fdfbccb`. Associated exact-certificate source and tests are identified in the companion provenance file. No public deposit identifier is assigned here.