Auxiliary Registers and the Compact Obstruction: An Exact First-Order Parent for a Linear Helicity-2 Sector

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Abstract

A pure compact paired-ladder realization of the linear tensor gauge symmetry has no onsite or one-derivative neutral phase character. Its smooth magnetic potential therefore starts at fourth order in momentum and produces a $z = 2$ mode with ordinary electric inertia. We ask whether local auxiliary registers can evade that obstruction without reintroducing a tensor mass. We construct a twelve-component first-order parent containing six symmetric-tensor coordinates and six connection-like auxiliaries. The microscopic tensor block is only a naive gradient term, while the exact Schur complement is the lattice linearized-Einstein kernel. The extended local symmetry forbids a constant tensor mass, and, after the exact scalar constraint is imposed, the model has precisely two degenerate branches with $\omega = |\widehat{d}|$ and a finite auxiliary gap. Exact symbolic identities, a 180-point finite-parameter scan, and finite-$\mathbb{Z}_7$ real-space audits are reported. We also prove a limitation: if the compact Hamiltonian is a positive smooth function of exact invariant characters and the auxiliaries are gapped, the $O(k^2)$ contribution cancels in the Schur complement and the effective stiffness remains $O(k^4)$. Thus the successful construction is Palatini/Villain-like rather than a smooth positive compact-character phase. Finally, a doubled flavored $U(2)$ link admits a mixed determinant that breaks the diagonal color $U(1)$ while preserving an orthogonal Abelian charge, removing the simplest same-generator conflict with the tensor sector. A bounded deconfined finite-register many-body realization remains open.

1 Question and status

The preceding commutant audit established two facts. Exact linear tensor gauge symmetry forbids an analytic transverse-traceless mass, but the first pure compact phase characters occur at two derivatives. Squaring those curvatures produces

$$K_{\mathrm{compact}}(k) = O(k^4), \qquad \omega_{\mathrm{compact}}(k) = O(k^2), \tag{1}$$

not the desired linear branch. The question here is narrower and decidable:

> Can a finite number of local auxiliary registers generate the two-derivative Einstein kernel as a Schur complement, while preserving an exact local constraint algebra and a finite auxiliary gap?

The answer is *yes* at the exact constrained quadratic and modular-algebra levels. The answer is *no* within the narrower class of positive smooth compact invariant-character Hamiltonians with gapped auxiliaries. Those statements are compatible and their distinction is the main result of this report.

2 Exact first-order factorization

Let the orthonormal six-component tensor coordinate be

$$h = (h_{xx}, h_{yy}, h_{zz}, \sqrt{2}h_{xy}, \sqrt{2}h_{xz}, \sqrt{2}h_{yz})^T. \tag{2}$$

For a commuting lattice derivative vector $d = (d_x, d_y, d_z)$, define the gauge map

$$\delta h = D(d)\xi, \tag{3}$$

which is the orthonormal form of $\delta h_{ij} = d_i\xi_j + d_j\xi_i$. Define the divergence and gradient-of-trace maps

$$(Vh)_j = d_i h_{ij}, \qquad (Gh)_j = d_j h_{ii}, \tag{4}$$

and combine them as

$$L(d) = \begin{pmatrix} 2V - G \\ G \end{pmatrix}. \tag{5}$$

Exact symbolic reduction gives the nontrivial identity

$$\boxed{K_{\mathrm{E}}(d) = |d|^2 I_6 - \frac{1}{2}L(d)^T L(d),} \tag{6}$$

where K_{E} is exactly the independently enumerated lattice linearized-Einstein kernel. The identity is not fitted numerically.

Introduce six auxiliary coordinates y and the canonical parent

$$\boxed{M(d) = \begin{pmatrix} |d|^2 I_6 & L(d)^T \\ L(d) & 2I_6 \end{pmatrix}.} \tag{7}$$

Its extended gauge transformation is

$$\delta y = -\frac{1}{2}L(d)D(d)\xi. \tag{8}$$

The exact identities are

$$M(d)\begin{pmatrix} D(d) \\ -L(d)D(d)/2 \end{pmatrix} = 0, \tag{9}$$

$$|d|^2 I_6 - L^T(2I_6)^{-1}L = K_{\mathrm{E}}(d). \tag{10}$$

Thus the microscopic h–h block contains only a componentwise local gradient energy. The Einstein tensor structure is generated by the auxiliary cross coupling and elimination of the auxiliary block.

A finite exact-symmetry family is

$$M(a, m, \eta) = \begin{pmatrix} (a + \eta s)K_{\mathrm{E}} + \frac{m}{4}L^T L & \frac{m}{2}L^T \\ \frac{m}{2}L & mI_6 \end{pmatrix}, \qquad s = |d|^2, \tag{11}$$

whose Schur complement is $(a + \eta s)K_{\mathrm{E}}$. The special point $m = 2a$, $\eta = 0$ reduces its tensor block to asI_6 and therefore makes the first-order origin especially transparent.

3 Mass protection and physical spectrum

The auxiliary transformation in Eq. (8) starts at second derivative order. If a constant tensor mass block $M^{(0)}_{hh}$ were allowed, the first-order gauge condition would require

$$M^{(0)}_{hh} D_1(d) = 0 \qquad \text{for every } d. \tag{12}$$

The union of the images of $D_1(d)$ spans the full six-dimensional symmetric-tensor coordinate space. Consequently

$$\boxed{M^{(0)}_{hh} = 0.} \tag{13}$$

The exact extended symmetry therefore forbids an analytic local tensor mass.

At nonzero momentum we quotient the three exact gauge directions and impose the scalar constraint

$$C_0 h = (d_i d_j h_{ij} - d^2 h_{ii}) = 0. \tag{14}$$

On a TT state, $Vh = Gh = 0$, so $Lh = 0$. Equation (7) then gives two exactly degenerate branches,

$$\omega_+(d) = \omega_-(d) = |d|, \tag{15}$$

and the first auxiliary frequency is $\sqrt{2}$ in the canonical normalization. The numerical fit over the infrared sequence is

$$z = 1.0000000000021, \qquad R^2 = 1. \tag{16}$$

Both low eigenvectors have TT weight equal to one to machine precision.

We sampled 180 parameter triples with $a \in [0.25, 3]$, $m \in [0.35, 5]$, and a finite higher-derivative coefficient $\eta \in [-0.3, 0.8]$. Every constrained sample was stable. The fitted exponents obeyed

$$0.9999704 \le z \le 1.0000769, \tag{17}$$

while the smallest auxiliary potential eigenvalue was 0.3501388. No parameter was retuned with volume.

4 The scalar constraint is load-bearing

The same Einstein kernel has a negative conformal branch on the gauge quotient. If the scalar constraint is omitted, the lowest potential eigenvalue approaches $-|d|^2$. With the scalar constraint imposed exactly, the low spectrum is nonnegative and contains only the two TT branches before the auxiliary gap.

This qualification is central. The construction is not evidence that an arbitrary finite penalty dynamically suppresses the conformal mode. It is an exact constrained parent.

5 Why smooth positive compact auxiliaries still give $z = 2$

The complete first-order exact invariant character is

$$u = y + \frac{1}{2} Lh. \tag{18}$$

The pure-h first-order syzygy space is zero. Therefore every first-order invariant character is a linear combination of u. Let a positive smooth compact potential have a gapped quadratic Hessian $W_u > 0$ in these characters. Its leading contribution is

$$\frac{1}{2} u^T W_u u. \tag{19}$$

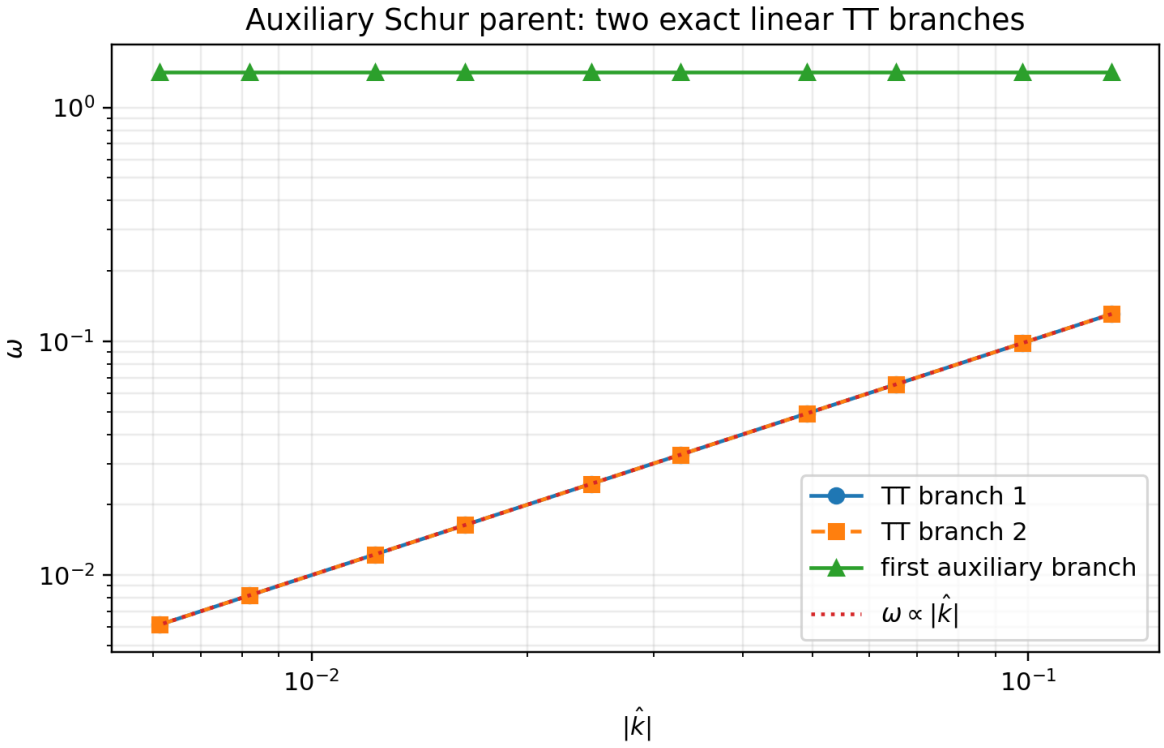

Figure 1: The two physical tensor branches are exactly linear and degenerate. The auxiliary sector remains separated by a finite gap.

Eliminating y cancels its entire $O(k^2)$ tensor stiffness. The remaining pure-h invariant characters begin with the curvature, which is $O(k^2 h)$. Their positive quadratic Hessian consequently gives

$$K_{\text{eff}}(k) = O(k^4), \qquad \omega(k) = O(k^2). \tag{20}$$

This is a Schur-complement statement, not merely a fit. Eighty random positive character Hessians yielded

$$\overline{z} = 1.9999989 \tag{21}$$

with a spread of order 10^{-5}.

Equation (20) does not forbid every finite-register realization. It excludes the positive smooth compact-character class with gapped auxiliaries. Signed Palatini-like forms, nonanalytic Villain or clock energies, nondynamical auxiliaries, and controlled noncompact limits remain outside its assumptions. The parent in Eq. (7) belongs to that broader constrained first-order class.

6 Exact finite-register algebra

We implemented the extended symmetry over $\mathbb{Z}_7$ on periodic cubic lattices. Each site carries six tensor and six auxiliary registers. A central local difference realizes D and L, and the finite-field inverse of two realizes Eq. (8). The modular quadratic form is exactly invariant under the local gauge shifts.

The scalar-gauge residual and matrix-symmetry residual also vanish exactly. This establishes a finite-register algebra and an exactly invariant modular quadratic phase. It does not establish that a bounded smooth clock Hamiltonian possesses a deconfined ground-state phase with the same harmonic kernel.

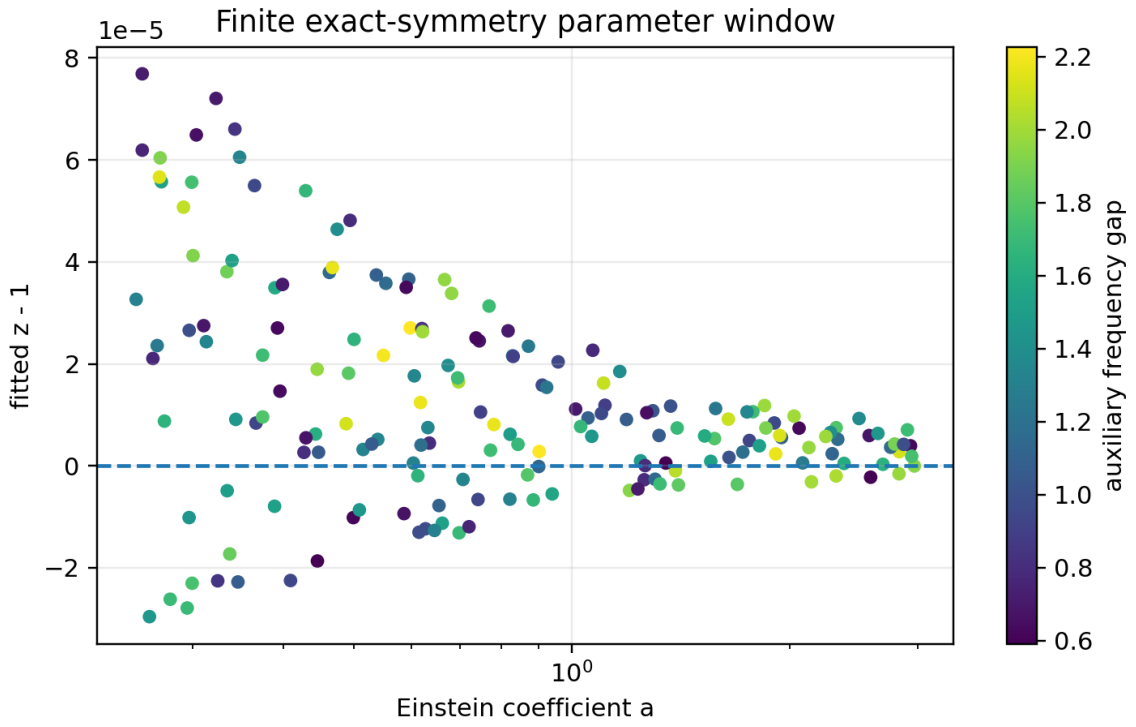

Figure 2: Deviation of the fitted exponent from one across the finite exact-symmetry parameter window. The color scale is the auxiliary frequency gap.

Table 1: Finite-$\mathbb{Z}_7$ real-space audits. Every listed residual is exactly zero modulo seven.

L	volume	register dimension	nonzero couplings	gauge residual	phase residual
3	27	324	3888	0	0
4	64	768	8064	0	0
5	125	1500	18000	0	0

7 An orthogonal-charge route around the determinant conflict

A single flavored $U(2)$ determinant carries charge two under its link electric generator and therefore cannot preserve that same generator as a tensor phase charge. Doubling the link gives an orthogonal combination. For two identical flavored links define

$$\mathcal{D}_{\mathrm{mix}} = \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}U_{+}^{\alpha\beta}U_{-}^{\gamma\delta}, \tag{22}$$

$$E_{\mathrm{color}} = E_{+} + E_{-}, \qquad E_{\mathrm{tensor}} = E_{+} - E_{-}. \tag{23}$$

Exact sparse arithmetic in the 36×36 one-link sectors, or dimension 1296 after doubling, gives

$$[\mathcal{D}_{\mathrm{mix}}, L_{+}^{a} + L_{-}^{a}] = 0, \tag{24}$$

$$[\mathcal{D}_{\mathrm{mix}}, R_{+}^{a} + R_{-}^{a}] = 0, \tag{25}$$

$$[E_{\mathrm{tensor}}, \mathcal{D}_{\mathrm{mix}}] = 0, \tag{26}$$

$$[E_{\mathrm{color}}, \mathcal{D}_{\mathrm{mix}}] = 2\mathcal{D}_{\mathrm{mix}}. \tag{27}$$

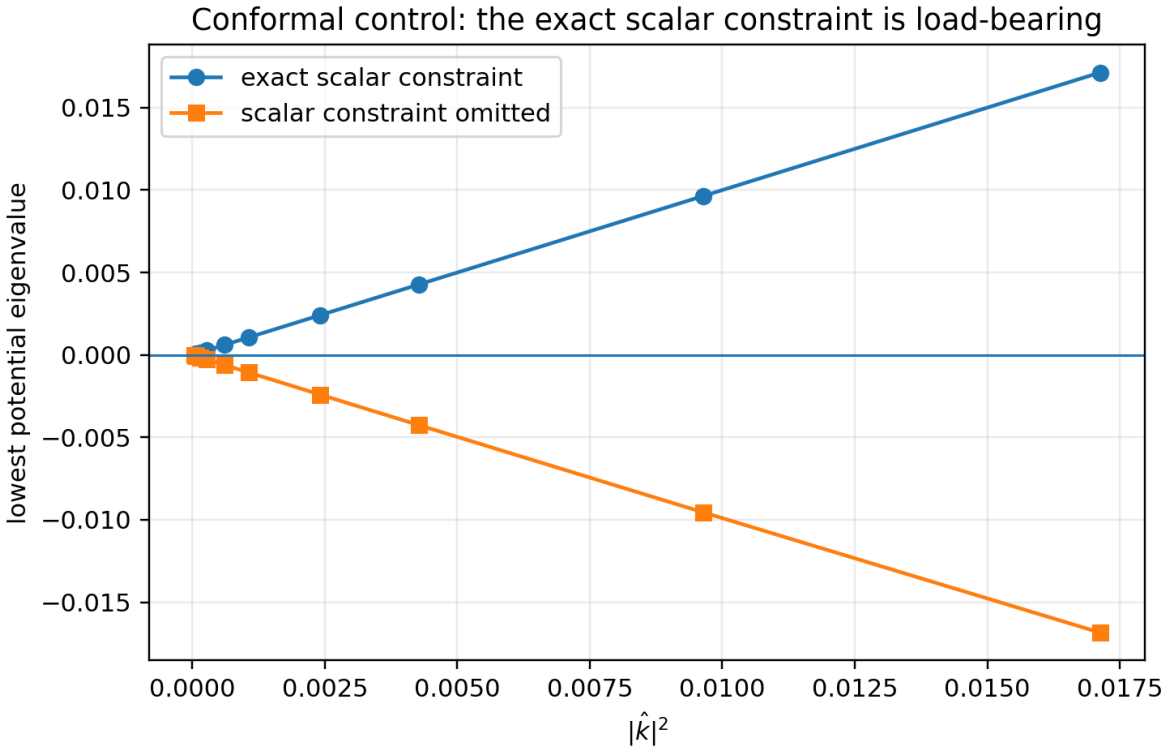

Figure 3: The conformal control. The first-order parent is not a stable unconstrained oscillator Hamiltonian; exact removal of the scalar branch is structural.

The largest diagonal-$SU(2)$ residual is 2.22×10^{-16}. The Hermitian term $\mathcal{D}_{\text{mix}} + \mathcal{D}_{\text{mix}}^{\dagger}$ preserves E_{tensor} exactly and breaks E_{color} with commutator Frobenius norm 113.1371.

This removes the simplest same-generator obstruction: the broken color $U(1)$ and preserved tensor $U(1)$ can be orthogonal combinations. It does not by itself lock the doubled links into a complete interacting shared-register phase.

8 Verdict and next decisive calculation

The exact conclusion is

> A local first-order auxiliary parent generates the lattice Einstein kernel as a Schur complement and supports a mass-forbidden, two-polarization $z = 1$ tensor sector after an exact scalar constraint. Its extended symmetry has an exact finite-p register realization, and a doubled flavored link supplies an algebraic $SU(2)$ determinant route that preserves an orthogonal tensor charge. However, positive smooth compact character Hamiltonians with gapped auxiliaries remain $z = 2$, and a bounded deconfined finite-register phase has not yet been demonstrated.

The next decisive calculation is therefore no longer another quadratic fit. It is to construct a bounded local Villain/clock Hamiltonian implementing the modular first-order parent and the exact scalar stabilizer, then establish from that same many-body Hamiltonian:

1. a finite constraint-defect gap;
2. two $z = 1$ TT poles as volume and local dimension increase;
3. no generated tensor mass under every exact-symmetry-preserving local perturbation;
4. deconfinement over a finite coupling interval rather than at one clock critical point;

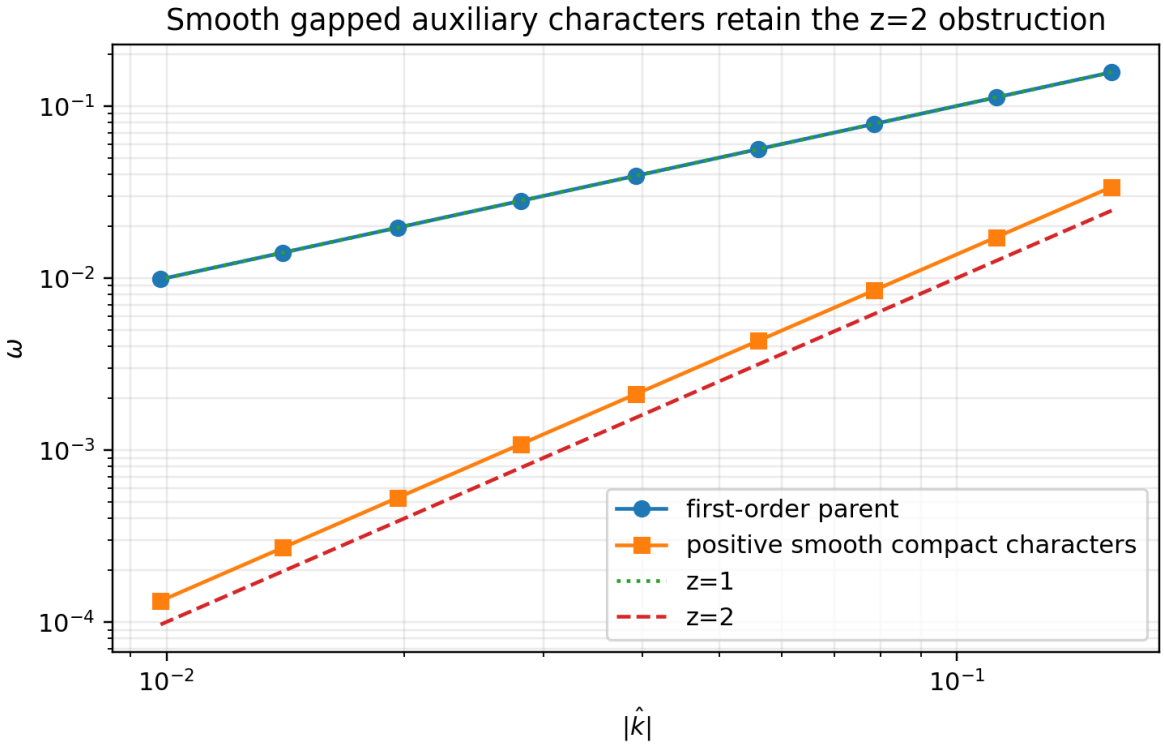

Figure 4: A representative positive smooth compact-character model remains $z = 2$, while the signed/constrained first-order parent is $z = 1$.

5. compatibility with the mixed determinant and mobile $SU(2)$ matter.

Reproducibility

The authoritative script is `code/run_auxiliary_audit.py`. It writes the complete JSON ledger, regenerates all figures, and reports the SHA-256 digest of the numerical output.

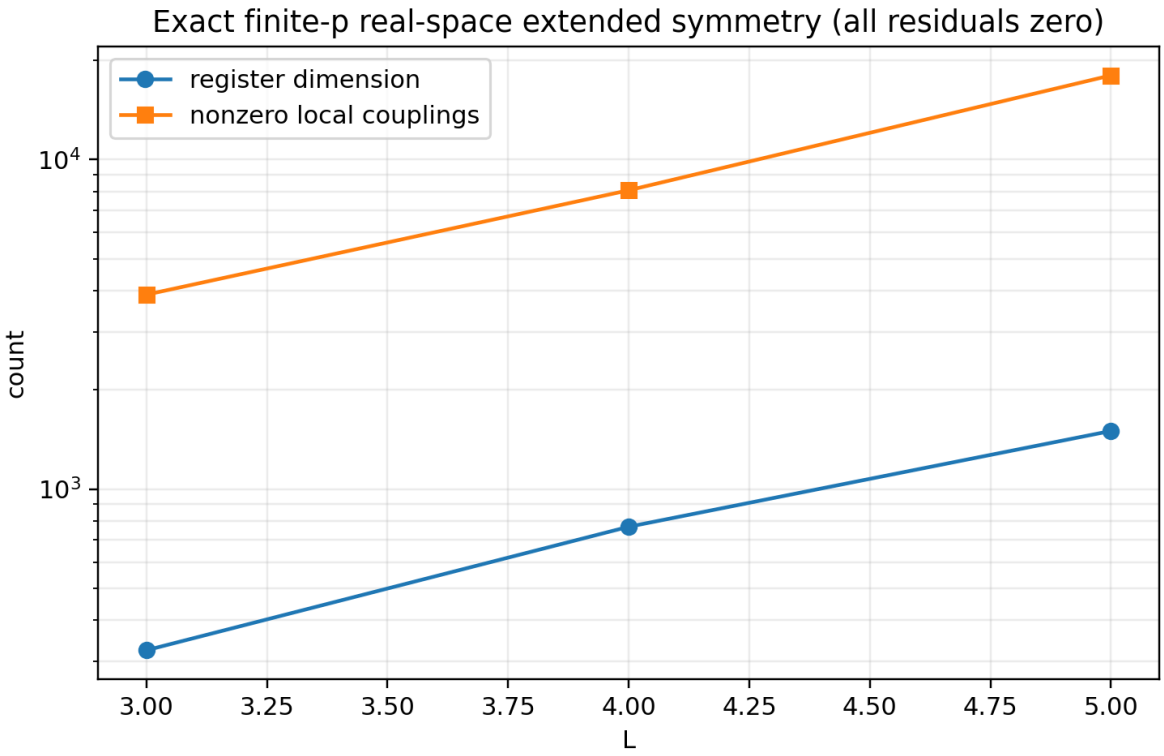

Figure 5: Size of the exact finite-register audit. All modular algebra residuals are zero for every size shown.

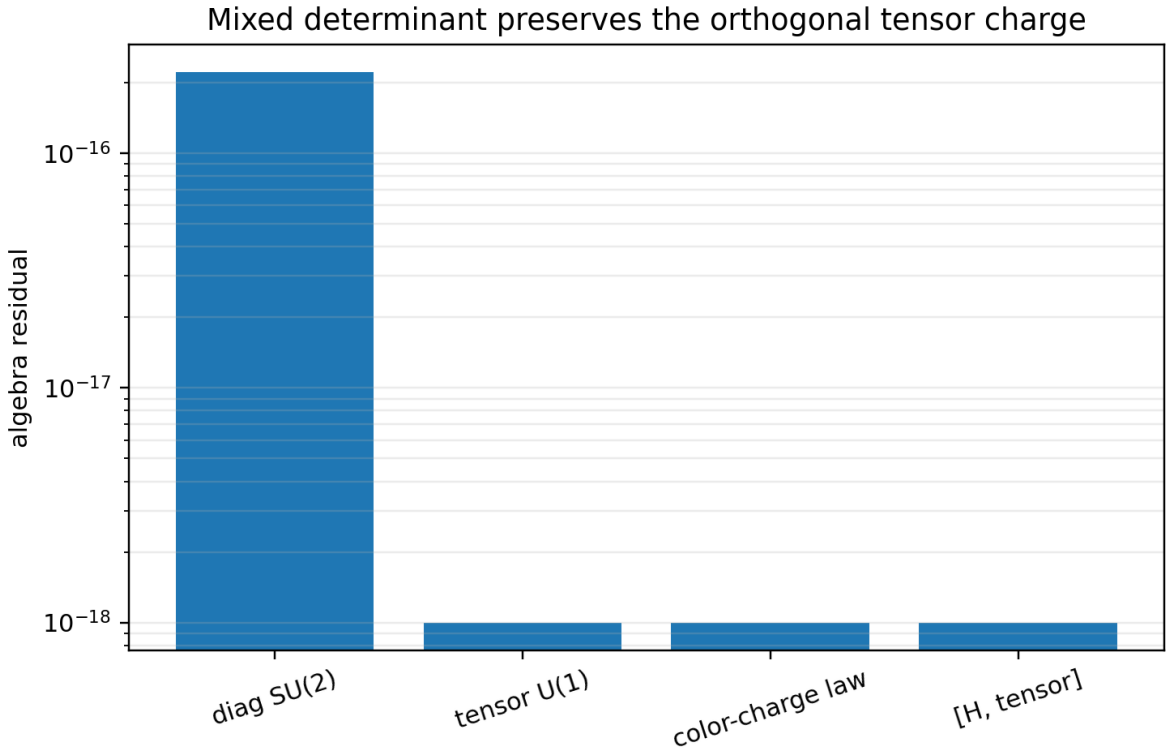

Figure 6: Algebra residuals for the mixed determinant. The diagonal color $U(1)$ is deliberately broken and is therefore not included as a residual bar.